On the Application of Quantum Perturbation Theory to Gravitational Interactions.

Bryce Seligman DeWitt
Harvard University, Cambridge, Massachusetts
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ABSTRACT

The formalism preliminary to a quantum perturbation treatment of the interaction of wave fields with gravitation is here developed. Since spinor fields are of importance, a resumé is given of Pauli's treatment of spinors in general coordinates, involving the introduction of generalized Dirac operators Γ^{μ}. The essential points of the Einstein-Mie theory are outlined, and spin angular momentum is discussed from the general coordinate viewpoint with the aid of the generalized orthogonal group. The commutation law for covariant differentiation is obtained for arbitrary fields. The symmetric stress tensor can be constructed either from the canonical energy-momentum tensor together with the spin angular momentum tensor or directly through variation of the metric tensor.

The concepts of energy, momentum and spin angular momentum, and hence the Hamiltonian formalism itself, can be introduced for the gravitational field only with respect to a "background space" which has a flat metric of no physical geometrical significance. In the background space only Lorentz transformations have immediate invariant significance, and general coordinate transformations appear as "gauge transformations" in the gravitational and accompanying matter fields. Only the total integrated energy, momentum and spin angular momentum quantities are invariant under these "gauge transformations."

The perturbation treatment of gravitational interactions is based on a variant of the weak field approximation method. The true metric $g_{\mu\nu}$ is an ordinary field quantity and is expressed in the form $\delta_{\mu\nu} + h_{\mu\nu}$. Other field quantities are expanded in ascending powers of $h_{\mu\nu}$, successive terms describing quantum transitions of increasing complexity and being of decreasing orders of magnitude. Explicit examination of the self-interactions of the gravitational field (due to its non-linearity) will never be made in subsequent calculations. To the first order, the matter-field equations can be derived, via the correspondence $p_{\mu} \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x_{\mu}}$, by considering the ponderomotive equations of a particle in a gravitational field.

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ON THE APPLICATION OF QUANTUM PERTURBATION THEORY TO GRAVITATIONAL INTERACTIONS. I. THE EINSTEIN-MIE THEORY, SPINORS, THE BACKGROUND SPACE AND THE APPROXIMATION METHOD.*

Bryce Seligman DeWitt†
Harvard University, Cambridge, Massachusetts

1. Introduction.

Recently, attention has been directed toward the problem of formulating the field equations of Einstein's gravitational theory in a canonical Hamiltonian manner, with the object in mind of effecting an eventual quantization of the gravitational field.[1,2] It is therefore of interest to examine

[1] P. G. Bergmann, Phys. Rev. 75, 680 (1949); P. G. Bergmann and J. H. M. Brunings, Rev. Mod. Phys. 21, 480 (1949); and papers to be published.
[2] F. A. E. Pirani and A. E. Schild, paper to be published.

a characteristic few of those aspects of a quantum theory of gravitation which make themselves particularly evident when the theory is applied to specific physical problems. It will be the purpose of this and a subsequent paper to undertake such an examination from the viewpoint of perturbation theory.

Of special interest is the examination of the extent to which the infinity-difficulties of ordinary quantum field theories are carried over into a theory based on the principle of general covariance. Bergmann[1] and his collaborators hope to avoid the infinities by treating interacting matter in terms of point singularities in the field, after the manner of Einstein, Infeld and Hoffman.[3] This point of view will not be adopted in the present

[3] A. Einstein, L. Infeld, and B. Hoffman, Ann. Math. 39, 65 (1938); also A. Einstein and L. Infeld, Ann. Math. 41, 455 (1940); and L. Infeld and P. R. Wallace, Phys. Rev. 57, 797 (1940).

papers. The Einstein-Mie theory, in which matter (as well as electromagnetic

* This and a paper to follow contain some results of a Ph. D. dissertation submitted to Harvard University (1950).
† NRC Postdoctoral Fellow, now at the Institute for Advanced Study, Princeton, N. J.

radiation) is described in terms of a continuous stress tensor, will be used instead. The stress tensor will, itself, be constructed out of matter or radiation field variables which are distinct from the gravitational field variables $g_{\mu\nu}$. It will be shown that the familiar infinities continue to occur in such a quantized gravitational theory just as they always have in the usual field theories of electrodynamics and the nucleus, but with at least one unexpected exception. The gravitational self-energy of an electromagnetic quantum will be found to vanish identically (to the second order in a perturbation approximation), this fact being independent of any gauge-invariance or renormalization arguments.

Another problem which eventually should be carefully examined in a quantum treatment of the gravitational field is that of the ambiguity, which arises from the strongly non-linear character of the theory, in the ordering of various non-commuting factors in a given term of the field equations. This problem will, however, be carefully sidestepped for the present by adopting a procedure derived from a point of view which may be described as follows: In the logical development of the Einstein theory, the principle of the equivalence of gravitational and inertial fields leads to the concept that all fields interact with the gravitational field. The mathematical formulation of this concept, via differential geometry and the principle of general covariance, leads to a non-linear field theory. The non-linearity, exemplified in the field equations, may be regarded as an expression of the fact that even the gravitational field itself is subject to gravitational effects, i.e., interacts with itself. In order to avoid the factor-ordering ambiguity, in the formulation of the theory adopted in these papers, the gravitational self-interaction effects will be isolated and then ignored insofar as any quantitative discussion of them is concerned. This isolation is easily accomplished when a perturbation philosophy is adopted. The linear (zero order) part of the gravitational Lagrangian function is subtracted from the full Lagrangian.

The residue, which contains all the self-interaction effects and troublesome ambiguities, is treated as a perturbation but is never closely examined. This procedure results, of course, in a rather makeshift formulation of the theory, which is, however, sufficient for the problems to be considered.

The use of a makeshift formulation has also been prompted by a more pressing consideration. At the time of initial writing of these papers no one had yet explicitly constructed a gravitational Hamiltonian for the general theory.[4] A Hamiltonian formulation of the "linearized" gravitational field, however, is a simple matter and can be carried out by well-known methods. The possibility of an "interaction representation" for the theory has therefore been tacitly assumed, the imagined "interaction Hamiltonian density" being simply what is left over when the linear part is subtracted from the hypothetical "total Hamiltonian density." Detailed questions about the interaction of the gravitational field with itself are never asked, and that part of the interaction Hamiltonian density which describes the self-interactions is never closely examined. But that part which describes the interaction of the gravitational field with other fields can be, and is, explicitly constructed for each special case considered.

This first paper will be primarily devoted to a mathematical description of the Einstein-Mie theory. Various features of the theory of special subsequent importance, such as "G-gauge-invariance," will be carefully discussed.

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[4] F. A. E. Pirani and A. E. Schild, and independently, P. G. Bergmann and his co-workers (loc. cit.) have just recently succeeded in constructing the required Hamiltonian function. Their results will be published shortly. Although the methods of construction are different in the two cases, the results are believed to be equivalent. The author is indebted to Professor Schild for permission to read one of the manuscripts prior to publication. The formulation of Pirani and Schild follows a method due to Dirac: (The Dynamical Theory of Fields, Canadian Mathematical Congress, Univ. of British Columbia, August, 1949). It may eventually be of interest to reformulate some of the present calculations according to the rigorous Pirani-Schild-Dirac scheme.

The theory of spinors and a brief discussion of spin in general coordinates will be included, and the approximation method to be used in later calculations will be introduced. Finally, the opportunity will also be taken of collecting in one place a few elementary results which have been partially neglected in the literature.

In the second paper, the quantization of the free (linear) gravitational field and a covariant elimination of the longitudinal field-components will be carried out. Then gravitational self-interactions and interactions with other fields will be introduced, and interaction Hamiltonian densities will be explicitly constructed for special cases. Finally, various calculations will be carried out in the covariant style introduced by Schwinger and Tomonaga. A brief discussion of the results will follow each calculation.

2. The General Coordinate Formalism. Spinor Theory.

As usual, Latin indices will range over the values 1, 2, 3 and Greek indices over the values 1, 2, 3, 4. Use of an imaginary fourth coordinate ξ^4 will be made from the very beginning, although the real coordinate $\xi^0 = -i\,\xi^4$ will be introduced occasionally. It will be necessary to recall the definition of real and imaginary field quantities when an imaginary fourth coordinate is present. If $\mu\cdots$, $\nu\cdots$ are coordinate-related indices then the conjugate of a non-spinor field quantity $\mathfrak{F}^{\mu\cdots}{}_{\nu\cdots}$ is defined by

$$\overline{\mathfrak{F}^{\mu\cdots}{}_{\nu\cdots}} = (-)^q \; (\mathfrak{F}^{\mu\cdots}{}_{\nu\cdots})^* \tag{2.1}$$

where q is the number of times the index 4 occurs among the $\mu\cdots, \nu\cdots$ In the case of c-numbers the asterisk* denotes the ordinary complex conjugate; in the case of operators or matrices it denotes the Hermitian adjoint. $\mathfrak{F}^{\mu\cdots}{}_{\nu\cdots}$ is said to be real if $\overline{\mathfrak{F}^{\mu\cdots}{}_{\nu\cdots}} = \mathfrak{F}^{\mu\cdots}{}_{\nu\cdots}$, imaginary if $\overline{\mathfrak{F}^{\mu\cdots}{}_{\nu\cdots}} = -\,\mathfrak{F}^{\mu\cdots}{}_{\nu\cdots}$, and complex otherwise. The reality properties of a field quantity evidently remain invariant under any coordinate transformation, or under any ordinary or covariant differentiation. Thus, for example, the metric tensor, $g_{\mu\nu}$, with components g_{ij}, g_{44} real and g_{i4}, g_{4j} pure imaginary, is a real tensor. Similarly, the Kronecker tensor, the Christoffel symbols and the curvature tensor are all real, while the Levi-Civita tensor densities[5] are imaginary.

[5] The Levi-Civita symbol, $\epsilon_{\mu\nu\sigma\tau}$, which is antisymmetric in each pair of indices, with $\epsilon_{1234} = 1$, may be regarded either as a contravariant tensor density $\delta^{\mu\nu\sigma\tau}$ of weight 1 or as a covariant tensor density $\delta_{\mu\nu\sigma\tau}$ of weight -1.

The conjugate of a spinor quantity will be defined at a later point.

In studying the 4-dimensional space-time manifold into which the coordinates ξ^μ are introduced, one considers the group of all coordinate transformations of the form

$$\xi'^i = f^i(\xi^1, \xi^2, \xi^3, \xi^0), \quad \xi'^0 = f^0(\xi^1, \xi^2, \xi^3, \xi^0) \qquad (2.2)$$

where f^1, f^2, f^3, f^0 are real functions whose derivatives to all orders exist everywhere in the region of interest[6] and have non-vanishing Jacobian $\left|\frac{\partial \xi'}{\partial \xi}\right| \neq 0$.

[6] It is sufficient for practically all purposes to require differentiability only up to the fourth order.

Tensor densities transform under such coordinate transformations according to the familiar law

$$\mathcal{J}'^{\mu\cdots}{}_{\nu\cdots} = \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \cdots\cdots \frac{\partial \xi^\tau}{\partial \xi'^\nu} \cdots\cdots \left|\frac{\partial \xi}{\partial \xi'}\right|^W \mathcal{J}^{\sigma\cdots}{}_{\tau\cdots}, \qquad (2.3)$$

where W is the weight.

Covariant differentiation is defined by

$$\mathcal{J}^{\mu\cdots}{}_{\nu\cdots;\sigma} = \mathcal{J}^{\mu\cdots}{}_{\nu\cdots,\sigma} + \left\{{\mu \atop \alpha\sigma}\right\}\mathcal{J}^{\alpha\cdots}{}_{\nu\cdots} + \cdots\cdots + \qquad (2.4)$$
$$-\left\{{\alpha \atop \nu\sigma}\right\}\mathcal{J}^{\mu\cdots}{}_{\alpha\cdots} - \cdots\cdots - W\left\{{\alpha \atop \alpha\sigma}\right\}\mathcal{J}^{\mu\cdots}{}_{\nu\cdots},$$

where the comma denotes ordinary differentiation. The covariant derivatives of the metric tensor, Kronecker tensor, and Levi-Civita tensor densities all vanish. Covariant differentiation, like ordinary differentiation, is distributive over factors in a product, but indices induced by repeated covariant differentiation do not, in general, commute. For example, if ϕ_μ is a covariant vector , then

$$\phi_{\sigma;\mu\nu} - \phi_{\sigma;\nu\mu} = -R_{\mu\nu\sigma}{}^{\tau}\phi_\tau \tag{2.5}$$

where $R_{\mu\nu\sigma}{}^{\tau}$ is the Riemann-Christoffel curvature tensor.

$$R_{\mu\nu\sigma}{}^{\tau} = \left\{{\tau \atop \sigma\mu}\right\}_{,\nu} - \left\{{\tau \atop \sigma\nu}\right\}_{,\mu} + \left\{{\tau \atop \alpha\nu}\right\}\left\{{\alpha \atop \sigma\mu}\right\} - \left\{{\tau \atop \alpha\mu}\right\}\left\{{\alpha \atop \sigma\nu}\right\}. \tag{2.6}$$

In its covariant form the curvature tensor satisfies the following algebraic identities

$$\begin{aligned} R_{\mu\nu\sigma\tau} = -R_{\nu\mu\sigma\tau} = -R_{\mu\nu\tau\sigma} = R_{\sigma\tau\mu\nu} \\ R_{\mu\nu\sigma\tau} + R_{\sigma\mu\nu\tau} + R_{\nu\sigma\mu\tau} = 0 \end{aligned} \tag{2.7}$$

and the well known differential Bianchi identities

$$R_{\mu\nu\alpha\beta;\sigma} + R_{\sigma\mu\alpha\beta;\nu} + R_{\nu\sigma\alpha\beta;\mu} = 0 . \tag{2.8}$$

The curvature tensor has 20 algebraically independent components. By introducing a contracted tensor $R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha} = R_{\nu\mu}$, having 10 algebraically independent components, and a scalar $R = g^{\mu\nu} R_{\mu\nu}$, one may, on contracting the Bianchi identities obtain the four identities

$$G^{\mu\nu}{}_{;\nu} = 0 \qquad \text{where} \qquad G^{\mu\nu} = R^{\mu\nu} - 1/2 g^{\mu\nu} R. \tag{2.9}$$

Spinors are most conveniently introduced into the general coordinate formalism after the manner of Pauli.[7] Since Pauli's method is not generally

[7] Annalen der Physik, Bd. 18 (1933).

familiar, a resumé will be given. One begins by generalizing the concept of orthogonal transformation. Consider an arbitrary non-singular symmetric 4 x 4 matrix ($g^{\mu\nu}$). The transformation expressed by a matrix ($\alpha^{\mu}{}_{\nu}$) is said to be orthogonal with respect to $g^{\mu\nu}$, or to define a generalized rotation with respect to $g^{\mu\nu}$, if it leaves $g^{\mu\nu}$ invariant, i.e., if $\alpha^{\mu}{}_{\sigma}\, \alpha^{\nu}{}_{\tau}\, g^{\sigma\tau} = g^{\mu\nu}$.
This reduces to the ordinary definition when $g^{\mu\nu}$ is simply the Kronecker delta $\delta_{\mu\nu}$. An infinitesimal generalized rotation is expressed by a transforma-

tion of the form $\alpha^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu}$ where the $\epsilon^{\mu}{}_{\nu}$ are infinitesimals satisfying $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ or $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, $g^{\mu\nu}$ and its inverse $g_{\mu\nu}$ having been used, in the usual way, to raise and lower indices.

Corresponding to these generalized orthogonal transformations one may introduce generalized Dirac operators Γ^{μ} which satisfy the anticommutator relations

$$\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2g^{\mu\nu} . \qquad (2.10)$$

Such generalized operators may be constructed from the ordinary Dirac operators γ_{μ} by taking $\Gamma^{\mu} = \sqrt{g}^{\mu\nu}\gamma_{\nu}$ where $\sqrt{g}^{\mu\nu}$ denotes a square root of $g^{\mu\nu}$. Since the ordinary Dirac operators can be conversely obtained from the generalized operators (by taking the inverse to $\sqrt{g}^{\mu\nu}$), it is evident that the Γ^{μ} have the same group theoretical properties as the γ_{μ}, namely, I: There exists only one irreducible faithful matrix representation of the Γ^{μ} (which is 4-dimensional); II: If $[A, \Gamma^{\mu}] = 0$ for all μ, then A is a multiple of the unit operator; II: If $\{\Gamma'^{\mu}, \Gamma'^{\nu}\} = 2g^{\mu\nu}$ then there must exist an operator S such that $\Gamma'^{\mu} = S^{-1}\Gamma^{\mu}S$. The operator S may, without loss of generality, always be restricted to have unit determinant in the 4-dimensional matrix representation.

The Γ^{μ} are readily shown to satisfy the following identities

$$[\Gamma^{\mu}, [\Gamma^{\nu}, \Gamma^{\sigma}]] = 4(g^{\mu\nu}\Gamma^{\sigma} - g^{\mu\sigma}\Gamma^{\nu}) \qquad (2.11)$$

$$\{\Gamma^{\mu}, [\Gamma^{\nu}, \Gamma^{\sigma}]\} = 2/3[\Gamma^{\mu}, \Gamma^{\nu}, \Gamma^{\sigma}] . \qquad (2.12)$$

The triple-operator bracket expression is defined by $[A,B,C] = A[B,C] + B[C,A] + C[A,B]$ and is antisymmetric in each pair of operators. Using (2.11) one may show that the operator S corresponding to the infinitesimal rotation $\delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu}$ is given by

$$S = 1 + 1/8\, \epsilon_{\mu\nu}[\Gamma^{\mu}, \Gamma^{\nu}] . \qquad (2.13)$$

It is clear that a set of operators Γ^{μ} can be defined throughout all of space-time, which define the metric tensor in any coordinate system by means of a relation of the form (2.10). The coordinate-transformation properties of the Γ^μ will be those of a contravariant vector. The Γ^μ, instead of the metric tensor, may be regarded as the fundamental quantities which describe the geometrical properties of space-time. One difficulty presents itself immediately, however. We have not shown that the Γ^μ can be chosen so as to be differentiable throughout a given region of space-time. The construction of the Γ^μ by means of a square root $\sqrt{g}^{\mu\nu}$ certainly does not insure the required differentiability. For example, if $\sqrt{g}^{\mu\nu}$ is obtained by diagonalizing $g^{\mu\nu}$, extracting the root, and transforming back again, then under certain circumstances, such as when two eigenvalues are nearly equal, a very small change in $g^{\mu\nu}$ will produce a very large change in the matrix required to diagonalize it. This produces a correspondingly large change in $\sqrt{g}^{\mu\nu}$, and hence in the Γ^μ.

The situation is saved by using a different method of construction. It is readily seen that the contravariant metric tensor will be altered by an infinitesimal amount $\delta g^{\mu\nu}$ if the Γ^μ are altered by infinitesimal amounts given by

$$\delta \Gamma^\mu = 1/2 \; \Gamma_\nu \delta g^{\mu\nu} \tag{2.14}$$

If $\delta g^{\mu\nu}$ is always chosen to be an indefinitely differentiable infinitesimal contravariant tensor then the Γ^μ may be built up continuously from any given indefinitely differentiable set, e.g., from the ordinary Dirac operators. Since the space-time manifold is characterized as a non-positive-definite metric space with signature (+,+,+,-), by the requirement that it always be possible to introduce a coordinate system at any point, in which the metric tensor becomes simply the Kronecker delta (and the Γ^μ become ordinary Dirac operators) at that point, it is evident that not only will $g^{\mu\nu}$ and the Γ^μ be indefinitely differentiable, but also $g^{\mu\nu}$ can be real ($\overline{g^{\mu\nu}} = g^{\mu\nu}$) and non-singular at every stage of the construction (2.14).

It is to be noted that the Γ^μ are not uniquely determined by the above construction. Since expression (2.14) is not, in general, an exact differential, the final form for the Γ^μ will depend upon the path of integration taken in the 10-dimensional space of the $g^{\mu\nu}$.

The differentiability of the contravariant vector operator Γ^μ now having been demonstrated, its covariant derivative may be defined by considering the related vector operator $\Gamma'^\mu = \Gamma^\mu + \epsilon^\sigma(\Gamma^\mu{}_{,\sigma} + \left\{{\mu \atop \alpha\sigma}\right\}\Gamma^\alpha)$, where ϵ^σ is an arbitrary infinitesimal contravariant vector. We find $\frac{1}{2}\{\Gamma'^\mu, \Gamma'^\nu\} = g^{\mu\nu} + \epsilon^\sigma g^{\mu\nu}{}_{;\sigma} = g^{\mu\nu}$, which implies there must exist an operator of the form $S = 1 + \epsilon^\sigma \Omega_\sigma$, where Ω_σ is some covariant vector operator, such that $\Gamma'^\mu = S^{-1}\Gamma^\mu S$. Since ϵ^σ is arbitrary this in turn implies $\Gamma^\mu{}_{,\sigma} + \left\{{\mu \atop \alpha\sigma}\right\}\Gamma^\alpha = -[\Omega_\sigma, \Gamma^\mu]$. If we now define the covariant derivative of Γ^μ by

$$\Gamma^\mu{}_{;\sigma} \equiv \Gamma^\mu{}_{,\sigma} + \left\{{\mu \atop \alpha\sigma}\right\}\Gamma^\alpha + [\Omega_\sigma, \Gamma^\mu] \tag{2.15}$$

this derivative is seen to vanish, and the property of covariant differentiation, of being distributive over factors in a product, is maintained; e.g. $g^{\mu\nu}{}_{;\sigma} = 1/2\,\{\Gamma^\mu, \Gamma^\nu\}_{;\sigma} = 1/2\{\Gamma^\mu{}_{;\sigma}, \Gamma^\nu\} + 1/2\{\Gamma^\mu, \Gamma^\nu{}_{;\sigma}\} = 0$. The covariant form of (2.15) is

$$\Gamma_{\mu;\sigma} \equiv \Gamma_{\mu,\sigma} - \left\{{\alpha \atop \mu\sigma}\right\}\Gamma_\alpha + [\Omega_\sigma, \Gamma_\mu] = 0\,. \tag{2.16}$$

The operator Ω_σ may be shown to be indefinitely differentiable by constructing it in a manner parallel to the construction of Γ^μ as expressed by (2.14). In order to maintain the vanishing property of the covariant derivative of Γ^μ under the variation (2.14), we must have

$$\begin{aligned}
0 &= (\Gamma^\mu + \delta\Gamma^\mu)_{;\sigma} \\
&= (\Gamma^\mu + \delta\Gamma^\mu)_{,\sigma} + (\left\{{\mu \atop \alpha\sigma}\right\} + \delta\left\{{\mu \atop \alpha\sigma}\right\})(\Gamma^\alpha + \delta\Gamma^\alpha) + [\Omega_\sigma + \delta\Omega_\sigma, \Gamma^\mu + \delta\Gamma^\mu] \\
&= \Gamma^\mu{}_{;\sigma} + \tfrac{1}{2}\Gamma_{\nu,\sigma}\,\delta g^{\mu\nu} + \tfrac{1}{2}\Gamma_\nu\,\delta g^{\mu\nu}{}_{,\sigma} + \tfrac{1}{2}\left\{{\mu \atop \alpha\sigma}\right\}\Gamma_\nu\,\delta g^{\alpha\nu} + \delta\left\{{\mu \atop \alpha\sigma}\right\}g^{\alpha\nu}\Gamma_\nu \\
&\quad + 1/2\,[\Omega_\sigma, \Gamma_\nu]\,\delta g^{\mu\nu} + [\delta\Omega_\sigma, \Gamma^\mu] \\
&= 1/2\,\Gamma_\nu\,[\delta g^{\mu\nu}{}_{,\sigma} + \delta(\left\{{\mu \atop \alpha\sigma}\right\}g^{\alpha\nu} + \left\{{\nu \atop \alpha\sigma}\right\}g^{\mu\alpha})] + 1/2\,\Gamma_\nu\,[\delta\left\{{\mu \atop \alpha\sigma}\right\}g^{\alpha\nu} - \delta\left\{{\nu \atop \alpha\sigma}\right\}g^{\alpha\mu}] + [\delta\Omega_\sigma, \Gamma^\mu]
\end{aligned}$$

in which we have used the fact that $\Gamma_{\nu,\sigma}\,\delta g^{\mu\nu} + [\Omega_\sigma, \Gamma_\nu]\,\delta g^{\mu\nu} = \left\{{\alpha \atop \nu\sigma}\right\}\Gamma_\alpha\,\delta g^{\mu\nu} = \left\{{\nu \atop \alpha\sigma}\right\}\Gamma_\nu\,\delta g^{\mu\alpha}$. The first term in the last line above is simply $1/2\;\Gamma_\nu\,\delta(g^{\mu\nu}{}_{;\sigma})$ and vanishes. We are left with $[\delta\Omega_\sigma, \Gamma^\mu] = 1/2\,\Gamma_\nu[g^{\mu\alpha}\,\delta\left\{{\nu \atop \alpha\sigma}\right\} - g^{\nu\alpha}\,\delta\left\{{\mu \atop \alpha\sigma}\right\}]$, the solution of which is found, by use of (2.11), to be

$$\delta\Omega_\sigma = \frac{1}{16}\,[g_{\mu\alpha}\,\delta\{^{\alpha}_{\nu\sigma}\} - g_{\nu\alpha}\,\delta\{^{\alpha}_{\mu\sigma}\}]\,[\Gamma^\mu, \Gamma^\nu] + \delta c\,, \qquad (2.17)$$

where δc is an arbitrary infinitesimal multiple of the unit operator. Starting from a flat Kronecker metric, with $\Gamma^\mu = \gamma_\mu$ and $\Omega_\sigma = 0$, Ω_σ can evidently be built up continuously along with the Γ^μ, remaining indefinitely differentiable at every stage.

The method of construction indicated by (2.14) and (2.17) is, of course, not the most general method by which the Γ^μ and Ω_σ may be formed. The Γ^μ may, at any stage, be transformed by an indefinitely differentiable operator S. Writing $0 = S^{-1}\Gamma^\mu{}_{;\sigma}S = (S^{-1}\Gamma^\mu S)_{,\sigma} + \{^{\mu}_{\alpha\sigma}\}S^{-1}\Gamma^\alpha S + [(S^{-1}\Omega_\sigma S + S^{-1}S_{,\sigma}), S^{-1}\Gamma^\mu S]$, we see that Γ^μ and Ω_σ transform together according to

$$\Gamma'^\mu = S^{-1}\Gamma^\mu S, \qquad \Omega'_\sigma = S^{-1}\Omega_\sigma S + S^{-1}S_{,\sigma} \qquad (2.18)$$

Since, in the 4-dimensional matrix representation, $|S| = 1$ and <u>spur</u> $(S^{-1}S_{,\sigma}) = |S|^{-1}|S|_{,\sigma} = 0$, the spur of Ω_σ remains unchanged under the transformation (2.18). Removing the arbitrariness in (2.17) by setting $\delta c = 0$, we may, without loss of generality, require Ω_σ always to have everywhere a vanishing spur. The transformation (2.18) is said to define a rotation in spin-space.

Now, if the conjugate to equation (2.10) is taken in the form $2g^{\mu\nu} = 2\bar{g}^{\mu\nu} = \{\bar{\Gamma}^\nu, \bar{\Gamma}^\mu\} = \{-\bar{\Gamma}^\mu, -\bar{\Gamma}^\nu\}$, one is led to infer the existence of an operator A with the property[8]

$$-\bar{\Gamma}^\mu = A\,\Gamma^\mu A^{-1}\;. \qquad (2.19)$$

[8] The vector operators Γ^μ and Ω_σ are, in general, complex.

Equation (2.19) and its conjugate, in turn, imply the commutability of the operator[9] $A^{*-1}A$ with the Γ^μ. This means $A^{*-1}A$ is equal to some multiple c of

[9] The asterisk* will always denote the Hermitian adjoint, although the more usual notation is †.

the unit operator. Thus $A = cA^* = cc^*A$, and c is seen to have the form

$c = e^{i\frac{\theta}{2}}$. A may, without loss of generality, be multiplied by $e^{-i\frac{\theta}{2}}$, thereby making it Hermitian. Under spin-space rotations A is readily seen to transform according to

$$A' = S^{*}AS, \tag{2.20}$$

and its Hermitian character remains undisturbed. If the Γ^{μ} are built up via (2.14) from a fixed set of ordinary Dirac operators γ_{μ}, in a representation in which the latter are unitary, and hence also Hermitian, having spurs equal to zero and determinants equal to unity, then A may be chosen equal to $\lambda\gamma_4$ where λ is a real number ($\neq 0$). If λ is chosen equal to ± 1, then A has unit determinant, and, under the restriction $|S| = 1$, the unicity of this determinant is maintained in any representation. A is invariant under coordinate transformations and is indefinitely differentiable.

A spinor ψ, in the 4-dimensional matrix representation, is defined as a 1-column matrix which is independent of coordinate transformations, but which transforms under spin-space rotations S according to

$$\psi' = S^{-1}\psi\ . \tag{2.21}$$

The conjugate of ψ is defined by

$$\bar{\psi} = \psi^{*}A \tag{2.22}$$

and transforms according to

$$\bar{\psi}' = \psi^{*'}A' = \psi^{*}S^{*-1}S^{*}AS = \bar{\psi}S \tag{2.23}$$

Spinors may be combined with the Γ^{μ}, as in $\bar{\psi}\psi$, $i\bar{\psi}\Gamma^{\mu}\psi$, to form ordinary tensors which are invariant under spin-space rotations. A tensor of the form

$$(i)^{r}\,\bar{\psi}\,\Gamma^{\mu_1}\ldots\Gamma^{\mu_r}\,\psi \tag{2.24}$$

is readily shown to be real.

The covariant derivative of a spinor is defined by invoking the perpetual condition that covariant differentiation be distributive over factors in a product. Writing $(\bar{\psi}\Gamma_{\mu}\psi)_{;\sigma} = \bar{\psi}_{;\sigma}\Gamma_{\mu}\psi + \bar{\psi}\Gamma_{\mu}\psi_{;\sigma}$, expanding the left hand side according to (2.4), and using (2.16), one finds

$$\psi_{;\sigma} \equiv \psi_{,\sigma} + \Omega_{\sigma}\psi\ , \qquad \bar{\psi}_{;\sigma} = \bar{\psi}_{,\sigma} - \bar{\psi}\Omega_{\sigma}\ . \tag{2.25}$$

Defining

$$\psi^*_{;\sigma} \equiv \psi^*_{,\sigma} + \psi^* \bar{\Omega}_\sigma , \tag{2.26}$$

we may write

$$\bar{\psi}_{;\sigma} = (\psi^* A)_{;\sigma} = \psi^*_{;\sigma} A + \psi^* A_{;\sigma} = \psi^*_{,\sigma} A + \psi^* \bar{\Omega}_\sigma A + \psi^* A_{;\sigma}$$
$$= (\psi^* A)_{,\sigma} - \psi^* A \Omega_\sigma = \psi^*_{,\sigma} A + \psi^* A_{,\sigma} - \psi^* A \Omega_\sigma$$

which implies

$$A_{;\sigma} \equiv A_{,\sigma} - A\Omega_\sigma - \bar{\Omega}_\sigma A . \tag{2.27}$$

Expression (2.27) actually vanishes. For, taking the conjugate of equation (2.16), $0 = \bar{\Gamma}_{\mu;\sigma} = \bar{\Gamma}_{\mu,\sigma} - \{{}^{\alpha}_{\mu\sigma}\}\bar{\Gamma}_\alpha - [\bar{\Omega}_\sigma, \bar{\Gamma}_\mu]$, and using equations (2.18) and (2.19), we may infer $-\bar{\Omega}_\sigma = A\Omega_\sigma A^{-1} - A_{,\sigma}A^{-1}$. Multiplication of this equation on the right by A leads to the conclusion $A_{;\sigma} = 0$. Since A is Hermitian, this result means, for example, that

$$\overline{\bar{\psi}\Gamma^\mu \psi_{;\nu}} = \psi^*_{;\nu} \bar{\Gamma}^\mu A \psi = -\psi^*_{;\nu} A \Gamma^\mu \psi = -\bar{\psi}_{;\nu}\Gamma^\mu \psi . \tag{2.28}$$

The easily demonstrated relations $\psi'_{;\sigma} = S^{-1}\psi_{;\sigma}$ and $\bar{\psi}'_{;\sigma} = \bar{\psi}_{;\sigma} S$ imply that the expression $\bar{\psi}\Gamma^\mu\psi_{;\nu}$ is invariant under spin-space rotations and is therefore an ordinary tensor.

Spinors with tensor indices are readily introduced into the general coordinate formalism. For example, the vector-spinor ψ_μ which transforms as a covariant vector under coordinate transformations and as a spinor under spin-space rotations, has a covariant derivative given by

$$\psi_{\mu;\sigma} \equiv \psi_{,\sigma} - \{{}^{\alpha}_{\mu\sigma}\}\psi_\alpha + \Omega_\sigma \psi . \tag{2.29}$$

Repeated covariant differentiation may be defined in a similar manner, and the spinor analogue of (2.5) becomes

$$\psi_{;\mu\nu} - \psi_{;\nu\mu} = (\Omega_{\mu,\nu} - \Omega_{\nu,\mu} - [\Omega_\mu, \Omega_\nu])\psi . \tag{2.30}$$

This relation may be expressed in a different form by observing that

$$0 = (\Gamma^\tau{}_{;\mu})_{,\nu} + \{{}^{\tau}_{\alpha\nu}\}\Gamma^\alpha{}_{;\mu} - \{{}^{\alpha}_{\mu\nu}\}\Gamma^\tau{}_{;\alpha} + [\Omega_\nu, \Gamma^\tau{}_{;\mu}]$$
$$- (\Gamma^\tau{}_{;\nu})_{,\mu} - \{{}^{\tau}_{\alpha\mu}\}\Gamma^\alpha{}_{;\nu} + \{{}^{\alpha}_{\nu\mu}\}\Gamma^\tau{}_{;\alpha} - [\Omega_\mu, \Gamma^\tau{}_{;\nu}]$$
$$= R_{\mu\nu\sigma}{}^\tau \Gamma^\sigma + [\Omega_{\mu,\nu} - \Omega_{\nu,\mu} - [\Omega_\mu, \Omega_\nu], \Gamma^\tau]$$

Using (2.11), one finds as the solution of this equation,

$$\Omega_{\mu,\nu} - \Omega_{\nu,\mu} - [\Omega_\mu, \Omega_\nu] = -\frac{1}{8} R_{\mu\nu\sigma\tau} [\Gamma^\sigma, \Gamma^\tau] \qquad (2.31)$$

which yields

$$\psi_{;\mu\nu} - \psi_{;\nu\mu} = -\frac{1}{8} R_{\mu\nu\sigma\tau} [\Gamma^\sigma, \Gamma^\tau] \psi . \qquad (2.32)$$

Other miscellaneous identities satisfied by the Γ^μ, which may be proved with the aid of identities (2.7), (2.11) and (2,12), are the following.

$$R_{\mu\nu\sigma\tau} \Gamma^\mu \Gamma^\nu \Gamma^\sigma \Gamma^\tau = 2R, \qquad (2.33)$$

$$(\underline{\varepsilon}_{\mu\nu\sigma\tau} \Gamma^\mu \Gamma^\nu \Gamma^\sigma \Gamma^\tau)^2 = (4!)^2 g^{-1}, \qquad (2.34)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$.

3. The Variational Fommulation. Spin in General Coordinates.

The field equations of the Einstein-Mie theory of gravitational interactions are derivable from a Lagrangian density of the form

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M \tag{3.1}$$

where

$$\mathcal{L}_G = -\frac{1}{\beta} g^{1/2} R, \quad ^{10} \tag{3.2}$$

$$\mathcal{L}_M = \mathcal{L}_M (Q_i, Q_{i;\sigma}, g_{\mu\nu}). \tag{3.3}$$

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10 When an imaginary fourth coordinate is used, g is positive. Therefore $g^{1/2}$ rather than $(-g)^{1/2}$ is the appropriate metric scalar density in this expression.

Q_i are the matter field variables and β is a constant related to the gravitational constant (see section 5). Denoting space-time integrals of $\mathcal{L}$, $\mathcal{L}_G$, $\mathcal{L}_M$ respectively by L, L_G, L_M, we write the field equations in the variational form

$$\frac{\delta L_G}{\delta g_{\mu\nu}} + \frac{\delta L_M}{\delta g_{\mu\nu}} = 0 , \qquad \frac{\delta L_M}{\delta Q_i} = 0. \tag{3.4}$$

As is well known,

$$\frac{\delta L_G}{\delta g_{\mu\nu}} = \frac{1}{\beta} g^{1/2} G^{\mu\nu} \tag{3.5}$$

where $G^{\mu\nu}$ is given by (2.9). It should be noted that although $\mathcal{L}_M$ is written in (3.3) as if it depended on the $g_{\mu\nu}$, the dependence, in the case of spinor fields, is rather on the fundamental operators Γ^μ. The field equations are in this case to be obtained by considering arbitrary variations in the Γ^μ, the most general permissable form of which is given by

$$\delta \Gamma^\mu = 1/2\, \Gamma_\nu\, \delta g^{\mu\nu} \quad -[\eta, \Gamma^\mu] \tag{3.6}$$

where η is an arbitrary infinitesimal matrix with vanishing spur which generates an infinitesimal spin-space rotation $S = 1 + \eta$. Under the requirement that $\mathcal{L}_M$ be invariant under spin-space rotations as well as under general coordinate transformations, the variation in L_M produced by (3.6) will nevertheless have the simple form

$$L_M = 1/2 \int g^{1/2}\left[\Theta_M^{\mu\nu} \delta g_{\mu\nu} + \frac{\delta L_M}{\delta Q_i} \delta Q_i \right] d\xi^1 d\xi^2 d\xi^3 d\xi^0 \quad (3.7)$$

where the δQ_i are variations in the Q_i produced by a spin-space rotation $S = 1 - \eta$. The second term in the above integrand vanishes when the matter field equations are satisfied, and it is customary in the case of both spinor and non-spinor fields to write

$$\Theta_M^{\mu\nu} \equiv 2g^{-1/2} \frac{\delta L_M}{\delta g_{\mu\nu}} . \quad (3.8)$$

$\Theta_M^{\mu\nu}$ is the symmetric matter-stress-tensor which generates the gravitational field via the Einstein field equations

$$G^{\mu\nu} = -\frac{\beta}{2} \Theta_M^{\mu\nu} . \quad (3.9)$$

The conservation properties of the stress tensor are derived by the long well-known method of considering infinitesimal coordinate transformations of the form

$$\xi'^{\mu} = \xi^{\mu} - \delta\Lambda^{\mu} . \quad (3.10)$$

Under (3.10) the matter field variables transform according to

$$Q_i'(\xi') = Q_i(\xi) + F_{i\mu}{}^{j\nu} Q_j(\xi) \delta\Lambda^{\mu}{}_{,\nu} , \quad (3.11)$$

and their derivatives, according to

$$Q'_{i,\sigma}(\xi') = \frac{\partial \xi^{\tau}}{\partial \xi'^{\sigma}} \frac{\partial}{\partial \xi^{\tau}} Q_i'(\xi') = (\delta^{\tau}_{\sigma} + \delta\Lambda^{\tau}{}_{,\sigma})\left[Q_i(\xi) + F_{i\mu}{}^{j\nu} Q_j(\xi) \delta\Lambda^{\mu}{}_{,\nu} \right]_{,\tau} , \quad (3.12)$$

where the $F_{i\mu}{}^{j\nu}$ are constants characterizing the transformation properties of the Q_i. If the $\delta\Lambda^{\mu}$ are restricted so as to vanish on the surface of an arbitrary region Ω, then, with the introduction of the variations

$$\delta Q_i(\xi) = Q_i'(\xi) - Q_i(\xi) = Q_{i,\mu}(\xi) \delta\Lambda^{\mu} + F_{i\mu}{}^{j\nu} Q_j(\xi) \delta\Lambda^{\mu}{}_{,\nu} , \quad (3.13)$$

$$\delta Q_{i,\sigma}(\xi) = Q'_{i,\sigma}(\xi) - Q_{i,\sigma}(\xi) = \frac{\partial}{\partial \xi^{\sigma}} \delta Q_i(\xi) , \quad (3.14)$$

and, in particular,

$$\delta g_{\mu\nu} = g_{\mu\nu,\sigma}\,\delta\Lambda^{\sigma} + g_{\sigma\nu}\,\delta\Lambda^{\sigma}{}_{,\mu} + g_{\mu\sigma}\,\delta\Lambda^{\sigma}{}_{,\nu}\,, \tag{3.15}$$

we may write, in virtue of the invariance of $\mathcal{L}_M$,

$$0 = \delta\int_{\Omega}\mathcal{L}_M\,d\xi^1 d\xi^2 d\xi^3 d\xi^0 = \int_{\Omega}\left[\frac{\delta L_M}{\delta Q_i}\,\delta Q_i + \frac{\delta L_M}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu}\right] d\xi^1 d\xi^2 d\xi^3 d\xi^0$$

$$= -\int_{\Omega} g^{1/2}\,\Theta_{M\mu}{}^{\nu}{}_{;\nu}\,\delta\Lambda^{\mu}\,d\xi^1 d\xi^2 d\xi^3 d\xi^0\,,$$

in which the matter field equations $\frac{\delta L_M}{\delta Q_i} = 0$ have been used and suitable partial integrations have been carried out. The arbitrariness of Ω and the $\delta\Lambda^{\mu}$ implies

$$\Theta_M{}^{\mu\nu}{}_{;\nu} = 0 \tag{3.16}$$

The same procedure may be used, with $\mathcal{L}_G$ substituted for $\mathcal{L}_M$, to derive the contracted Bianchi identities (2.9).

The electromagnetic field and the description of electrically charged matter by means of complex field variables may be explicitly included in the general coordinate formalism by writing the total Lagrangian density in the form $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_E + \mathcal{L}_M$, where

$$\mathcal{L}_E = -\frac{1}{16\pi}\,g^{1/2}\,F_{\mu\nu}F^{\mu\nu} \tag{3.17}$$

$$\mathcal{L}_M = \mathcal{L}_M(Q_i, \bar{Q}_i, D_\sigma Q_i, \bar{D}_\sigma \bar{Q}_i, g_{\mu\nu}) \tag{3.18}$$

$F_{\mu\nu} \equiv A_{\nu;\mu} - A_{\mu;\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$ where A_μ is the covariant electromagnetic potential vector. $\bar{Q}_i$ denotes the conjugate of Q_i, and the operators D_σ, $\bar{D}_\sigma$ are defined by

$$D_\sigma Q_i \equiv Q_{i;\sigma} - i\epsilon A_\sigma Q_i\,, \qquad \bar{D}_\sigma \bar{Q}_i = \bar{Q}_{i;\sigma} + i\epsilon A_\sigma \bar{Q}_i\,. \tag{3.19}$$

The constant ϵ is determined by quantization of the theory, to have the value $\frac{e}{\hbar c}$. In $\mathcal{L}_M$, the field variables Q_i will occur in products of the form $\bar{Q}_i Q_j$, $\bar{Q}_i D_\mu Q_j$, or $\bar{D}_\mu \bar{Q}_i D_\nu Q_j$, and the requirement that $\mathcal{L}_M$ be gauge-invariant leads to the gauge transformation laws

$$A'_\mu = A_\mu + \Lambda_{;\mu} = A_\mu + \Lambda_{,\mu}\,, \qquad Q'_i = e^{i\epsilon\Lambda}\,Q_i\,, \qquad \bar{Q}'_i = e^{-i\epsilon\Lambda}\,\bar{Q}_i\,, \tag{3.20}$$

Λ being the scalar gauge parameter.

The electromagnetic field equations, $\frac{\delta L_E}{\delta A_\mu} + \frac{\delta L_M}{\delta A_\mu} = 0$, have the explicit form

$$F^{\mu\nu}{}_{;\nu} = 4\pi S^{\mu} \tag{3.21}$$

where S^{μ} is the contravariant current vector.[11]

$$S^{\mu} \equiv \frac{\delta L_M}{\delta A_{\mu}} \equiv -i\epsilon\left[\frac{\partial \mathcal{L}_M}{\partial(D_{\mu}Q_i)}\; Q_i - \bar{Q}_i\; \frac{\partial \mathcal{L}_M}{\partial(\bar{D}_{\mu}\bar{Q}_i)}\right]. \tag{3.22}$$

[11] In quantum electrodynamics, the vector $j^{\mu} = cS^{\mu}$ is more frequently used.

The charge conservation law

$$S^{\mu}{}_{;\mu} = 0 \tag{3.23}$$

is derived by considering the variation induced in $\mathcal{L}_M$ by an infinitesimal gauge transformation and making use of the matter field equations and the gauge invariance of $\mathcal{L}_M$. Since $F^{\mu\nu}{}_{;\nu\mu} = R_{\mu\nu}F^{\mu\nu} = 0$, this law is compatible with (3.21). By considering the variation induced in L_M by the transformation (3.10) and making use of the (3.23), the matter field equations, and the explicit form

$$\delta A_{\mu} = A_{\mu,\sigma}\,\delta\Lambda^{\sigma} + A_{\sigma}\,\delta\Lambda^{\sigma}{}_{,\mu}, \tag{3.24}$$

one obtains the further relation

$$\Theta_M{}^{\mu\nu}{}_{;\nu} = F^{\mu}{}_{\nu}S^{\nu}. \tag{3.25}$$

In the construction of the symmetric matter-stress-tensor by variation of the metric tensor, the spin properties of the matter field remain completely hidden. The symmetric stress tensor may, however, also be constructed from a canonical energy-momentum tensor by a procedure which is analagous to that employed in Minkowski coordinates in ordinary flat-space field theory and which explicitly involves the spin properties of the matter field. Use is made of the invariance, at any given point, of the Lagrangian density under generalized orthogonal transformations (see section 2). Let the value $g^{\mu\nu}(\xi)$, of the contravariant metric tensor at the point ξ, define the generalized orthogonal group in question. Under the infinitesimal rotation $\delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu}$ the matter field variables will suffer a transformation of the form

$$Q_i' = Q_i + 1/2\,\epsilon_{\mu\nu}S^{\mu\nu}{}_i{}^j Q_j \tag{3.26}$$

where the $S^{\mu\nu}{}_i{}^j$ are coefficients characteristic of the fields Q_i, being related to the constants $F_{i\mu}{}^{j\nu}$ of (3.11) by

$$S^{\mu\nu}{}_i{}^j = g^{\nu\sigma}F_{i\sigma}{}^{j\mu} - g^{\mu\sigma}F_{i\sigma}{}^{j\nu} \tag{3.27}$$

The covariant derivative of Q_i will transform according to

$$Q'_{i;\sigma} = Q_{i;\sigma} + \epsilon_{\sigma}{}^{\tau}Q_{i;\tau} + 1/2\,\epsilon_{\mu\nu}S^{\mu\nu}{}_i{}^j Q_{j;\sigma} \tag{3.28}$$

and we may, at the point ξ, write

$$0 = \mathcal{L}_M(Q_i', Q'_{i;\sigma}, g_{\mu\nu}) - \mathcal{L}_M(Q_i, Q_{i;\sigma}, g_{\mu\nu})$$

$$= -1/2\,\epsilon_{\mu\nu}\left[S^{\mu\nu}{}_i{}^j Q_j \frac{\delta L_M}{\delta Q_i} + g^{\frac{1}{2}}\overset{s}{M}_M{}^{\mu\nu\sigma}{}_{;\sigma} + g^{\mu\sigma}Q_{i;\sigma}\frac{\partial \mathcal{L}_M}{\partial Q_{i;\nu}} - g^{\nu\sigma}Q_{i;\sigma}\frac{\partial \mathcal{L}_M}{\partial Q_{i;\mu}}\right]$$

where
$$\overset{s}{M}_M{}^{\mu\nu\sigma} = -g^{-1/2}S^{\mu\nu}{}_i{}^j \frac{\partial \mathcal{L}_M}{\partial Q_{i;\sigma}} Q_j \tag{3.29}$$

Since $\frac{\delta L_M}{\delta Q_i} = 0$ and the $\epsilon_{\mu\nu}$ are arbitrary, we have

$$\overset{s}{M}_M{}^{\mu\nu\sigma}{}_{;\sigma} = -g^{-1/2}\left[g^{\mu\sigma} Q_{i;\sigma} \frac{\partial \mathcal{L}_M}{\partial Q_{i;\nu}} - g^{\nu\sigma} Q_{i;\sigma} \frac{\partial \mathcal{L}_M}{\partial Q_{i;\mu}} \right] . \qquad (3.30)$$

(3.30), being a covariant equation valid at an arbitrary point, is valid everywhere. $\overset{s}{M}_M{}^{\mu\nu\sigma}$ is known as the spin angular momentum tensor of the matter field.

In the case of a covariant vector field ϕ_μ, equation (3.26) has the explicit form

$$\phi'_\sigma = \phi_\sigma + 1/2\, \epsilon_{\mu\nu} S^{\mu\nu}{}_\sigma{}^\tau \phi_\tau , \quad S^{\mu\nu}{}_\sigma{}^\tau = \delta^\mu_\sigma g^{\nu\tau} - \delta^\nu_\sigma g^{\mu\tau} \qquad (3.31)$$

In the case of spinor fields the vector Γ^μ, rather than the spinors themselves, is the transformed quantity. One is always at liberty, however, to effect a spin space rotation, namely, the inverse to (2.13), which restores Γ^μ to its original form. Spinors then transform according to

$$\left.\begin{aligned} \psi' &= \psi + 1/2\, \epsilon_{\mu\nu} S^{\mu\nu} \psi \\ \bar\psi' &= \bar\psi - 1/2\, \epsilon_{\mu\nu} \bar\psi S^{\mu\nu} \end{aligned}\right\} , \qquad S^{\mu\nu} = 1/4\, [\, \Gamma^\mu , \Gamma^\nu \,] \qquad (3.32)$$

By comparing equations (3.31) and (3.32) respectively with equations (2.5) and (2.32), one is led to infer the general validity, for all tensor and spinor fields, of the following equation

$$Q_{i;\mu\nu} - Q_{i;\nu\mu} = -1/2\, R_{\mu\nu\sigma\tau} S^{\sigma\tau}{}_i{}^j Q_j \ . \qquad (3.33)$$

Using this equation, we may write

$$0 \equiv \mathcal{L}_{M;\mu} - \mathcal{L}_{M;\mu} = \mathcal{L}_{M;\mu} - \frac{\partial \mathcal{L}_M}{\partial Q_i} Q_{i;\mu} - \frac{\partial \mathcal{L}_M}{\partial Q_{i;\nu}} Q_{i;\nu\mu}$$

$$= \mathcal{L}_{M;\mu} - \frac{\delta L_M}{\delta Q_i} Q_{i;\mu} - \left(\frac{\partial \mathcal{L}_M}{\partial Q_{i;\nu}} Q_{i;\mu} \right)_{;\nu} - 1/2\, \frac{\partial \mathcal{L}_M}{\partial Q_{i;\nu}} R_{\mu\nu\sigma\tau} S^{\sigma\tau}{}_i{}^j Q_j$$

$$= g^{1/2} \left[T_{M\mu}{}^\nu{}_{;\nu} + 1/2 R_{\mu\nu\sigma\tau} \overset{s}{M}_M{}^{\sigma\tau\nu} \right] \qquad (3.34)$$

where $$T_{M\mu}{}^\nu = g^{-1/2} \left[-Q_{i;\mu} \frac{\partial \mathcal{L}_M}{\partial Q_{i;\nu}} + \delta^\nu_\mu \mathcal{L}_M \right] . \qquad (3.35)$$

$T_{M\mu}{}^\nu$ is the canonical energy-momentum tensor. The interaction of the spin angular momentum with the gravitational field (in the form of the curvature tensor) is seen to prevent this tensor from satisfying the same conservation law that it satisfies in ordinary flat-space theory.

The symmetric stress tensor is constructed from $T_{M\mu}{}^\nu$ and $\overset{s}{M}_M{}^{\mu\nu\sigma}$ in the familiar manner

$$\Theta_M{}^{\mu\nu} = 1/2\, (T_M{}^{\mu\nu} + T_M{}^{\nu\mu}) - 1/2 (\overset{s}{M}_M{}^{\sigma\mu\nu} + \overset{s}{M}_M{}^{\sigma\nu\mu})_{;\sigma} \ . \qquad (3.36)$$

Noting that equation (3.30) may be written in the form

$$\overset{s}{M}_M{}^{\mu\nu\sigma}{}_{;\sigma} = T_M{}^{\mu\nu} - T_M{}^{\nu\mu} \qquad (3.37)$$

and making use of equation (3.34) and the identities (2.7) satisfied by the curvature tensor, one may rederive the conservation law (3.16) directly from (3.36).

We shall close this section by displaying three specific examples of the most familiar matter fields.

A. The Real Scalar Meson Field.

The Lagrangian density is

$$\mathcal{L}_M = -1/2 g^{1/2}(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \kappa^2\phi^2) , \quad (\phi_{,\mu} \equiv \phi_{;\mu}) . \tag{3.38}$$

The field equation is the generalized Klein-Gordon equation

$$g^{\mu\nu}\phi_{;\mu\nu} - \kappa^2\phi = 0. \tag{3.39}$$

The symmetric and canonical energy-momentum tensors are identical.

$$\Theta_{M\mu\nu} = T_{M\mu\nu} = \phi_{,\mu}\phi_{,\nu} - 1/2 g_{\mu\nu}(g^{\sigma\tau}\phi_{,\sigma}\phi_{,\tau} + \kappa^2\phi^2). \tag{3.40}$$

B. The Dirac Field.

The Lagrangian density is

$$\mathcal{L}_M = -\frac{\hbar c}{2} g^{1/2}(\bar{\psi}\Gamma^\mu\psi_{;\mu} - \bar{\psi}_{;\mu}\Gamma^\mu\psi + 2\kappa\bar{\psi}\psi), \tag{3.41}$$

with the corresponding field equation

$$\Gamma^\mu\psi_{;\mu} + \kappa\psi = 0 \qquad (\text{or} \quad \bar{\psi}_{;\mu}\Gamma^\mu - \kappa\bar{\psi} = 0) \tag{3.42}$$

Multiplying (3.42) on the left by Γ^ν , taking the covariant divergence, and using equations (2.32) and (2.33), one obtains

$$\begin{aligned} 0 &= g^{\mu\nu}\psi_{;\mu\nu} - \kappa^2\psi + 1/32 R_{\mu\nu\sigma\tau} [\Gamma^\mu, \Gamma^\nu][\Gamma^\sigma, \Gamma^\tau]\psi \\ &= g^{\mu\nu}\psi_{;\mu\nu} - \kappa^2\psi + 1/4 R\psi \end{aligned} \tag{3.43}$$

The curvature of space-time evidently prevents the Dirac field from exactly satisfying the generalized Klein-Gordon equation.

Since the value of the Lagrangian density is zero in virtue of the field equations (3.42), the canonical energy momentum tensor is given simply by

$$T_{M\mu}{}^{\nu} = g^{-1/2}\left[-\bar{\psi}_{;\mu}\frac{\partial\mathcal{L}_M}{\partial\bar{\psi}_{;\nu}} - \frac{\partial\mathcal{L}_M}{\partial\psi_{;\nu}}\psi_{;\mu}\right] = \frac{\hbar c}{2}\left(\bar{\psi}\Gamma^\nu\psi_{;\mu} - \bar{\psi}_{;\mu}\Gamma^\nu\psi\right) \tag{3.44}$$

the spin angular momentum tensor by

$$\overset{s}{M}_M{}^{\mu\nu\sigma} = -g^{-1/2}\left[\frac{\partial\mathcal{L}_M}{\partial\psi_{;\sigma}} S^{\mu\nu}\psi - \bar{\psi}S^{\mu\nu}\frac{\partial\mathcal{L}_M}{\partial\bar{\psi}_{;\sigma}}\right] = \frac{\hbar c}{8}\bar{\psi}\{\Gamma^\sigma, [\Gamma^\mu, \Gamma^\nu]\}\psi$$

$$= \frac{\hbar c}{12} \bar{\psi} \, [\Gamma^\mu, \Gamma^\nu, \Gamma^\sigma] \quad , \tag{3.45}$$

and the current vector by

$$S^\mu = -i\epsilon \left[\frac{\partial \mathcal{L}_M}{\partial \psi_{;\mu}} \psi - \bar{\psi} \frac{\partial \mathcal{L}_M}{\partial \bar{\psi}_{;\mu}} \right] = ie \bar{\psi} \Gamma^\mu \psi \, . \tag{3.46}$$

Since $\overset{s}{M}_M{}^{\sigma\mu\nu} + \overset{s}{M}_M{}^{\sigma\nu\mu} = 0$, the symmetric stress tensor is simply

$$\Theta_{M\mu\nu} = 1/2(T_{M\mu\nu} + T_{M\nu\mu}) \tag{3.47}$$

This form may also be derived by varying the Γ^μ. It is sufficient to consider a variation of the form (2.14). We have, using (2.17),

$$\delta L_M = -\frac{\hbar c}{2} \int g^{-1/2} \left[g^{-1/2} \mathcal{L}_M \delta g^{1/2} + \bar{\psi} \delta\Gamma^\mu \psi_{;\mu} + \bar{\psi}\Gamma^\mu \delta\Omega_\mu \psi - \bar{\psi}_{;\mu} \delta\Gamma^\mu \psi + \bar{\psi} \delta\Omega_\mu \Gamma^\mu \psi \right] d\xi^1 d\xi^2 d\xi^3 d\xi^0$$

$$= -\frac{\hbar c}{2} \int g^{1/2} \left[\bar{\psi}\Gamma_\nu \psi_{;\mu} - \bar{\psi}_{;\mu}\Gamma_\nu \psi + \bar{\psi}\Gamma_\mu \psi_{;\nu} - \bar{\psi}_{;\nu}\Gamma_\mu \psi \right] \delta g^{\mu\nu} \, d\xi^1 d\xi^2 d\xi^3 d\xi^0$$

$$- \frac{\hbar c}{2} \int g^{1/2} \left[g^{-\frac{1}{2}} \mathcal{L}_M \delta g^{\frac{1}{2}} + \tfrac{1}{16} \left(g_{\alpha\gamma} \delta \left\{ {\gamma \atop \beta\mu} \right\} - g_{\beta\gamma} \delta \left\{ {\gamma \atop \alpha\mu} \right\} \right) \bar{\psi} \{ \Gamma^\mu, [\Gamma^\alpha, \Gamma^\beta] \} \psi \right] d\xi^1 d\xi^2 d\xi^3 d\xi^0$$

The second integral vanishes owing to the antisymmetry of $\{\Gamma^\mu, [\Gamma^\alpha, \Gamma^\beta]\}$, via (2.12), and the fact that $\mathcal{L}_M = 0$. We are left with

$$\Theta_{M\mu\nu} = -2g^{-1/2} \frac{\delta L_M}{\delta g^{\mu\nu}} = \frac{\hbar c}{4} \left(\bar{\psi}\Gamma_\nu \psi_{;\mu} - \bar{\psi}_{;\mu}\Gamma_\nu \psi + \bar{\psi}\Gamma_\mu \psi_{;\nu} - \bar{\psi}_{;\nu}\Gamma_\mu \psi \right) . \tag{3.48}$$

C. The Electromagnetic Field.

Although not normally regarded as a "matter field," the electromagnetic field has, from the viewpoint of the gravitational field, exactly the same footing as any matter field. The Lagrangian density is given by (3.17). The field equations have the form

$$0 = -F_\mu{}^\nu{}_{;\nu} = g^{\nu\sigma} (A_{\mu;\nu\sigma} - A_{\nu;\mu\sigma}) \tag{3.49}$$

When the gauge is chosen so that the generalized Lorentz condition $A^\mu{}_{;\mu} = 0$ is satisfied, the field equations may be written

$$\begin{aligned} 0 &= g^{\nu\sigma} (A_{\mu;\nu\sigma} - A_{\nu;\sigma\mu} + A_{\nu;\sigma\mu} - A_{\nu;\mu\sigma}) \\ &= g^{\nu\sigma} A_{\mu;\nu\sigma} - g^{\nu\sigma} R_{\sigma\mu\nu}{}^\tau A_\tau \\ &= g^{\nu\sigma} A_{\mu;\nu\sigma} + R_\mu{}^\nu A_\nu \end{aligned} \tag{3.50}$$

Here again the curvature of space-time prevents the field from exactly satisfying the generalized Klein-Gordon equation (in this case, for $\kappa = 0$). Equations (3.43) and (3.50) may be combined into the general form

$$g^{\mu\nu} Q_{i;\mu\nu} - \kappa^2 Q_i + \frac{1}{4S} S^{\mu\nu}{}_i{}^k S^{\sigma\tau}{}_k{}^j R_{\mu\nu\sigma\tau} Q_j = 0 \tag{3.51}$$

where S is the spin number of the field in question (S = 1/2 for Dirac field, S = 1 for electromatnetic field.) It may be inferred that equation (3.51) is generally valid for all fields Q_i having a pure spin number, i.e. corresponding to particles of only one spin variety, and hence satisfying certain supplementary conditions, such as the generalized Lorentz condition, which eliminate particles of lower spin number.

The symmetric stress tensor of the electromagnetic field is

$$\Theta_{E\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha} F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) . \tag{3.52}$$

4. Energy, Momentum, and Spin in the Gravitational Field. G-Gauge Transformations and the Background Space.

The energy, momentum, and spin angular momentum of the gravitational field are most conveniently defined in terms of the well-known first order differential Lagrangian

$$\begin{aligned} \mathcal{L}'_G &= -\frac{1}{\beta} g^{1/2} g^{\mu\nu}\left(\left\{{\beta \atop \mu\nu}\right\}\left\{{\alpha \atop \beta\alpha}\right\} - \left\{{\alpha \atop \beta\mu}\right\}\left\{{\beta \atop \alpha\nu}\right\}\right) \\ &= -\frac{1}{4\beta} g^{1/2} g^{\mu\nu} g^{\alpha\sigma} g^{\beta\tau}\left(2 g_{\mu\tau,\nu} g_{\alpha\sigma,\beta} - g_{\mu\nu,\tau} g_{\alpha\sigma,\beta} - 2 g_{\mu\sigma,\beta} g_{\nu\tau,\alpha} + g_{\beta\mu,\sigma} g_{\nu\tau,\alpha}\right) \end{aligned} \tag{4.1}$$

which is obtained from the Lagrangian (3.2) by discarding a divergence term. In terms of $\mathcal{L}'_G$ the tensor $G^{\mu\nu}$ is given by

$$G^{\mu\nu} = \beta g^{-1/2}\left[\frac{\partial \mathcal{L}'_G}{\partial g_{\mu\nu}} - \left(\frac{\partial \mathcal{L}'_G}{\partial g_{\mu\nu,\sigma}}\right)_{,\sigma}\right] . \tag{4.2}$$

The gravitational energy-momentum conservation law may be obtained by rewriting (3.34) in non-covariant form. First note that $T_M{}^{\mu\nu} = \Theta_M{}^{\mu\nu} + F_M{}^{\mu\nu\sigma}{}_{;\sigma}$ where

$$F_M{}^{\mu\nu\sigma} = 1/2(\overset{s}{M}_M{}^{\mu\nu\sigma} - \overset{s}{M}_M{}^{\mu\sigma\nu} - \overset{s}{M}_M{}^{\nu\sigma\mu}) = -F_M{}^{\mu\sigma\nu} \tag{4.3}$$

and that $g^{1/2}\left\{{\alpha \atop \mu\nu}\right\} F_{M\alpha}{}^{\nu\sigma}{}_{;\sigma} = (g^{1/2}\left\{{\alpha \atop \mu\nu}\right\} F_{M\alpha}{}^{\nu\sigma})_{;\sigma} - 1/2\, g^{1/2} R_{\nu\sigma\mu}{}^{\alpha} F_{M\alpha}{}^{\nu\sigma}$. Then write

$$\begin{aligned} 0 &= g^{1/2}\left(T_{M\mu}{}^{\nu}{}_{;\nu} + 1/2\, R_{\mu\nu\sigma\tau} \overset{s}{M}_M{}^{\sigma\tau\nu}\right) \\ &= (g^{1/2} T_{M\mu}{}^{\nu})_{,\nu} - g^{1/2}[\mu\nu,\alpha]\Theta_M{}^{\alpha\nu} - g^{1/2}\left\{{\alpha \atop \mu\nu}\right\} F_{M\alpha}{}^{\nu\sigma}{}_{;\sigma} + 1/2 g^{1/2} R_{\mu\nu\sigma\tau} \overset{s}{M}_M{}^{\sigma\tau\nu} \\ &= T'_{M\mu}{}^{\nu}{}_{,\nu} + \frac{1}{\beta} g^{1/2} g_{\nu\alpha,\mu} G^{\alpha\nu} + 1/2 g^{1/2} R_{\mu\nu\sigma\tau}(\overset{s}{M}_M{}^{\nu\sigma\tau} + \overset{s}{M}_M{}^{\tau\nu\sigma} + \overset{s}{M}_M{}^{\sigma\tau\nu}) \\ &= \left(T_{G\mu}{}^{\nu} + T'_{M\mu}{}^{\nu}\right)_{,\nu} \end{aligned} \tag{4.4}$$

where $T'_{M\mu}{}^{\nu} = g^{1/2}(T_{M\mu}{}^{\nu} - [\mu\beta,\alpha]\, F_M{}^{\alpha\beta\nu})$ (4.5)

and $T_{G\mu}{}^{\nu} = -g_{\alpha\beta,\mu} \dfrac{\partial \mathcal{L}'_G}{\partial g_{\alpha\beta,\nu}} + \delta^{\nu}_{\mu} \mathcal{L}'_G$ (4.6)

Equation (4.4) may be given physical significance by considering the case in which the gravitational and matter fields are confined to a limited region of space. The fields may then be pictured as being confined within a suitable tube in space-time. Choose any coordinate system which goes over into flat Minkowskian form outside of this tube. Then treat these coordinates as being everywhere Minkowskian. That is, inside the tube introduce a new metric, the flat Kronecker metric $\delta_{\mu\nu}$, which has nothing to do with the tensor $g_{\mu\nu}$ which describes the gravitational field. Let σ be any 3-dimensional unbounded surface which is space-like with respect to this new metric. Then introduce the integral

$$P_\mu = \frac{1}{c}\int_\sigma \left(T_{G\mu}{}^{\nu} + T'_{M\mu}{}^{\nu}\right) d\sigma_\nu \tag{4.7}$$

where the directed surface element $d\sigma_\nu$ is defined with respect to the new metric. P_μ evidently transforms like a 4-vector under Lorentz transformations and, in virtue of (4.4), is independent of the surface σ. A much more important property of P_μ is its invariance under general coordinate transformations which take place inside the tube containing the fields. This invariance has long been known.[12] Its demonstration is rendered easy by considering the

[12] A. Einstein, Berl. Ber. (1918), p. 488.

value of P_μ at two different surfaces σ_1 and σ_2. Let its value at these two surfaces be respectively $\overset{1}{P}_\mu$ and $\overset{2}{P}_\mu$ before the interior coordinate transformation, and $\overset{1}{P}{}'_\mu$ and $\overset{2}{P}{}'_\mu$ afterwards. Introduce now a second interior coordinate transformation to a set of coordinates which coincide, up to derivatives of the second order, with the original coordinates at σ_1

and with the transformed coordinates at σ_2. Denote the corresponding values of P_μ in this case by $\overset{1}{P}{}''_\mu$ and $\overset{2}{P}{}''_\mu$. By construction we have $\overset{1}{P}{}''_\mu = \overset{1}{P}_\mu$ and $\overset{2}{P}{}''_\mu = \overset{2}{P}{}'_\mu$, and, in virtue of (4.4), of course, $\overset{1}{P}_\mu = \overset{2}{P}_\mu$, $\overset{1}{P}{}'_\mu = \overset{2}{P}{}'_\mu$ and $\overset{1}{P}{}''_\mu = \overset{2}{P}{}''_\mu$. The result is now evident.

The flat metric $\delta_{\mu\nu}$ which is introduced for the construction of P_μ may be regarded as the metric of a <u>background space</u>[13] which shares the coordinates ξ^μ with <u>real space</u>. With respect to the background space the components $g_{\mu\nu}$ of the <u>true</u> metric of <u>real space</u> are simply the field variables of the gravitational field. From the viewpoint of the background space only Lorentz transformations have invariant significance, and $T_{G\mu}{}^\nu$ and $T'_{M\mu}{}^\nu$ may be regarded as ordinary tensors, and P_μ as an ordinary 4-vector. $T_{G\mu}{}^\nu$ is the canonical energy-momentum tensor of the gravitational field, and P_μ is the energy momentum 4-vector of the combined gravitational and matter fields. Equation (4.4) expresses the law of conservation of energy and momentum.

From the point of view of the background space a general coordinate transformation may be regarded as producing a <u>gauge transformation</u> in the gravitational and accompanying matter fields. We shall call such a gauge transformation a <u>gravitational gauge transformation</u>, or, simply, a <u>G-gauge transformation</u>. Equations (3.13 - 3.15) then become the defining equations of an infinitesimal G-gauge transformation, the displacement $\delta\Lambda^\mu$ being the <u>G-gauge parameter</u>. The concept of general covariance, which applies to <u>real space</u>, is, in the background space, replaced by the concept of G-gauge invariance. The Lagrangian functions $\mathcal{L}'_G$ and $\mathcal{L}_M$ are altered, under G-gauge transformations, only by divergences. Hence the field equations are G-gauge invariant. Furthermore,

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[13] Pirani and Schild (<u>loc.cit.</u>) use the term "amorphous space" to express the same idea.

although the tensors $T_{G\mu}{}^{\nu}$ and $T'_{M\mu}{}^{\nu}$ are not themselves G-gauge invariant, the vector P_μ is.

A spin angular momentum tensor for the gravitational field may be introduced by the usual method, of which the procedure outlined in section 3, for the matter field, is merely a covariant generalization. The invariance of $\mathcal{L}'_G$ under Lorentz transformations leads to the equation

$$\overset{s}{M}_{G\mu\nu\sigma,\sigma} = T_{G\mu}{}^{\nu} - T_{G\nu}{}^{\mu} - \frac{2}{\beta} g^{1/2} G_\mu{}^\nu + \frac{2}{\beta} g^{1/2} G_\nu{}^\mu$$

$$= T_{G\mu}{}^{\nu} - T_{G\nu}{}^{\mu} + g^{1/2}\Theta_{M\mu}{}^{\nu} - g^{1/2}\Theta_{M\nu}{}^{\mu} \qquad (4.8)$$

where $$\overset{s}{M}_{G\mu\nu\sigma} = -2\left[\frac{\partial \mathcal{L}'_G}{\partial g_{\mu\tau,\sigma}} g_{\nu\tau} - \frac{\partial \mathcal{L}'_G}{\partial g_{\nu\tau,\sigma}} g_{\mu\tau}\right]. \qquad (4.9)$$

All the indices on the gravitational spin angular momentum tensor $\overset{s}{M}_{G\mu\nu\sigma}$ are written in the lower position since it has significance only with respect to Lorentz transformations in the background space. A symmetric stress tensor

$$\Theta_{G\mu\nu} = 1/2(T_{G\mu}{}^{\nu} + T_{G\nu}{}^{\mu}) - 1/2(\overset{s}{M}_{G\sigma\mu\nu} + \overset{s}{M}_{G\sigma\nu\mu})_{,\sigma} \qquad (4.10)$$

may likewise be defined for the gravitational field. Now, if, in the derivation of the conservation law (4.4), we had taken equation (3.16) as our starting point instead of equation (3.34), we would have obtained the result

$$\left(T_{G\mu}{}^{\nu} + g^{1/2}\Theta_{M\mu}{}^{\nu}\right)_{,\nu} = 0 \qquad (4.11)$$

Hence, defining $$\Theta'_{M\mu\nu} = \tfrac{1}{2} g^{1/2}(\Theta_{M\mu}{}^{\nu} + \Theta_{M\nu}{}^{\mu}) \qquad (4.12)$$

we may also write the conservation law

$$\left(\Theta_{G\mu\nu} + \Theta'_{M\mu\nu}\right)_{,\nu} = 0. \qquad (4.13)$$

The energy momentum 4-vector P_μ may be defined in terms of the combined symmetric stress tensor $\Theta_{\mu\nu} (= \Theta_{G\mu\nu} + \Theta_{M\mu\nu})$ just as well as in terms of the combined canonical tensor $T_{\mu\nu}$ $(= T_{G\mu}{}^{\nu} + T'_{M\mu}{}^{\nu})$, for the latter may also be written in the form

$$T_{\mu\nu} = -g_{\alpha\beta,\mu}\frac{\partial \mathcal{L}'}{\partial g_{\alpha\beta,\nu}} - Q_{i,\mu}\frac{\partial \mathcal{L}'}{\partial Q_{i,\nu}} + \delta_{\mu\nu}\mathcal{L}', \quad \mathcal{L}' = \mathcal{L}'_G + \mathcal{L}_M \qquad (4.14)$$

and the former in the form

$$\Theta_{\mu\nu} = T_{\mu\nu} - F_{\mu\nu\sigma,\sigma}, \qquad F_{\mu\nu\sigma} + F_{\mu\sigma\nu} = 0 \qquad (4.15)$$

where $F_{\mu\nu\sigma}$ is a tensor which depends on the spin properties of the combined fields. Since the integral $\int_\sigma F_{\mu\nu\sigma,\sigma}\, d\sigma_\nu$ vanishes on account of the theorem $\int_\sigma f_{,\mu}\, d\sigma_\nu = \int_\sigma f_{,\nu}\, d\sigma_\mu$ and the antisymmetry of $F_{\mu\nu\sigma}$ in its last two indices, we have $P_\mu = \frac{1}{c}\int_\sigma T_{\mu\nu}\, d\sigma_\nu = \frac{1}{c}\int_\sigma \Theta_{\mu\nu}\, d\sigma_\nu$. By a similar argument

$$P_\mu = \frac{1}{c}\int_\sigma (T_{G\mu}{}^{\nu} + g^{1/2}\Theta_{M\mu}{}^{\nu})\, d\sigma_\nu .$$

The combined spin-orbital angular momentum tensor is defined by

$$M_{\mu\nu\sigma} = \xi^\mu \Theta_{\nu\sigma} - \xi^\nu \Theta_{\mu\sigma} \qquad (4.16)$$

and the total angular momentum tensor by

$$J_{\mu\nu} = \frac{1}{c}\int_\sigma M_{\mu\nu\sigma}\, d\sigma_\sigma = \frac{1}{c}\int_\sigma (\overset{L}{M}_{\mu\nu\sigma} + \overset{S}{M}_{\mu\nu\sigma})\, d\sigma_\sigma \qquad (4.17)$$

where $\overset{L}{M}_{\mu\nu\sigma} = \xi^\mu T_{\nu\sigma} - \xi^\nu T_{\mu\sigma}$. (4.18)

($\overset{L}{M}_{\mu\nu\sigma}$ is the canonical orbital angular momentum tensor.) $J_{\mu\nu}$ is independent of σ because of the conservation law

$$M_{\mu\nu\sigma,\sigma} = 0. \qquad (4.19)$$

and, by an argument identical with that given for P_μ, its G-gauge invariance is readily made evident.

Although P_μ and $J_{\mu\nu}$ are G-gauge invariant, the stress tensor $\Theta_{\mu\nu}$, in spite of its symmetry, is not. This fact corresponds to a result obtained by Fierz.[14] The free gravitational field (i.e. free from interaction even with

[14] Fierz, Helv. Phys. Acta, 12, 3 (1939).

itself) is a massless spin-2 field, and Fierz has shown that the symmetric stress tensor of any massless field of spin greater than 1 is not gauge invariant even though gauge invariance is maintained for the integrated quantities P_μ and $J_{\mu\nu}$. This lack of gauge invariance is related to a lack of positive definiteness in the energy density of higher spin fields, whether they be massless or not, which was also shown by Fierz. Since, for the gravitational field, only the integrated quantities are G-gauge invariant, a precise characterization of the distribution of energy, momentum, or angular momentum in the gravitational field is impossible. For example, a non-vanishing stress tensor may be created out of an initially vanishing one simply by effecting a G-gauge transformation. This implies also that the sign of the energy density can, at any point, be altered so as to become positive or negative at will, by changing the G-gauge. Physical significance can be attached, therefore, only to the integrated quantities.

The G-gauge invariant viewpoint and the mathematical presence of the background space tend to obscure the existence of real space and the geometrical interpretation of the theory. Real space is even further obscured in the quantum theory. For there, since the metric tensor becomes a quantum operator, real space itself becomes quantized in a peculiar sort of way. These facts, together with the necessity for introducing the background space in order to construct a Hamiltonian formalism for effecting this quantization, might incline one to renounce the reality of real space in favor of the reality of the background space. But one cannot ascribe reality to the background space. In the first place a G-gauge transformation produces what would, with respect to the background space, be a very observable change, for example, in a very real quantity such as the current density or the electromagnetic field. And secondly, as we have seen, even the symmetric stress tensor cannot be used to lend reality to the background space by yielding a precise characterization of the distribution of energy, momentum and angular momentum in the gravitational field.

5. The Approximation Method.

For the purpose of calculating specific gravitational effects we shall introduce a variant of the well known "weak-field" approximation method. We shall assume that it is possible to introduce a coordinate system into the space-time manifold, with respect to which the metric tensor takes the form

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \tag{5.1}$$

where the $h_{\mu\nu}$ are small compared to unity. The $h_{\mu\nu}$ measure the deviation of space-time from flatness. Instead of following the more customary procedure of expressing this deviation as a series expansion in powers of a small parameter, the $h_{\mu\nu}$ will, for the time being, be allowed to denote the full gravitational potentials.

Since all subsequent discussion will be carried on in the G-gauge invariant language of the background space, we shall replace the coordinate symbols ξ^{μ} by the more common symbols x_{μ}. The index is written in the lower position on the x_{μ} since the x_{μ} will only be subjected to Lorentz transformations. Since $\delta_{\mu\nu}$ is Lorentz invariant, $h_{\mu\nu}$ is an ordinary symmetric tensor in the background space. When expressed in terms of $h_{\mu\nu}$, the G-gauge transformation law (3.15) takes the form

$$h'_{\mu\nu} = h_{\mu\nu} + \delta\Lambda_{\mu,\nu} + \delta\Lambda_{\nu,\mu} + h_{\mu\nu,\sigma}\delta\Lambda_{\sigma} + h_{\mu\sigma}\delta\Lambda_{\sigma,\nu} + h_{\nu\sigma}\delta\Lambda_{\sigma,\mu}\,, \tag{5.2}$$

in which the index on the G-gauge parameter has been lowered for the sake of uniformity.

The Lagrangian function for the combined gravitational and matter fields will now be expanded in the form

$$\mathcal{L}' = \overset{\circ}{\mathcal{L}}{}'_G + \mathcal{L}'^{\,\text{self}}_G + \overset{\circ}{\mathcal{L}}_M + \overset{1}{\mathcal{L}}_M + \overset{2}{\mathcal{L}}_M + \cdots\cdots \tag{5.3}$$

where

$$\left.\begin{aligned} \overset{\circ}{\mathcal{L}}{}'_G &= \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}'_G}{\partial g_{\alpha\beta,\mu}\,\partial g_{\gamma\delta,\nu}}\right)_0 h_{\alpha\beta,\mu}\, h_{\gamma\delta,\nu} \\ \mathcal{L}'^{\,\text{self}}_G &= \mathcal{L}'_G - \overset{\circ}{\mathcal{L}}{}'_G \end{aligned}\right\} \tag{5.4}$$

$$\overset{\circ}{\mathcal{L}}_M = (\mathcal{L}_M)_0$$

$$\left.\begin{aligned} \overset{1}{\mathcal{L}}_M &= \left(\frac{\partial \mathcal{L}_M}{\partial g_{\alpha\beta}}\right)_0 h_{\alpha\beta} + \left(\frac{\partial \mathcal{L}_M}{\partial g_{\alpha\beta,\mu}}\right)_0 h_{\alpha\beta,\mu} \\ \overset{2}{\mathcal{L}}_M &= \frac{1}{2!}\left(\frac{\partial^2 \mathcal{L}_M}{\partial g_{\alpha\beta}\,\partial g_{\gamma\delta}}\right)_0 h_{\alpha\beta} h_{\gamma\delta} + \left(\frac{\partial^2 \mathcal{L}_M}{\partial g_{\alpha\beta}\,\partial g_{\gamma\delta,\mu}}\right)_0 h_{\alpha\beta} h_{\gamma\delta,\mu} \\ &\quad + \frac{1}{2!}\left(\frac{\partial^2 \mathcal{L}_M}{\partial g_{\alpha\beta,\mu}\,\partial g_{\gamma\delta,\nu}}\right)_0 h_{\alpha\beta,\mu} h_{\gamma\delta,\nu} \end{aligned}\right\} \quad (5.5)$$

etc.

The subscript 0 indicates that the quantity inside the parentheses is to be evaluated by setting $g_{\mu\nu} = \delta_{\mu\nu}$ and $g_{\mu\nu,\sigma} = 0$.

Directly into this approximation scheme will be fitted a standard perturbation method based on the weakness of gravitational coupling, as measured by the smallness of the constant β . Quantities will be classified as to order of magnitude depending on the power of $\beta^{\frac{1}{2}}$ which they involve. The gravitational potentials $h_{\mu\nu}$ are evidently of the order of magnitude of $\beta^{\frac{1}{2}}$. Quantities may therefore equivalently be classified as to order depending on the power of the gravitational potentials which they involve. This gives the explanation for the superscripts appearing over the terms in the expansion (5.3).

For many purposes it is convenient to introduce another set of field variables $\phi_{\mu\nu}$ given by

$$\phi_{\mu\nu} = h_{\mu\nu} - 1/2\, \delta_{\mu\nu}\, h_{\sigma\sigma} \qquad (5.6)$$

This change in variables corresponds, in first order, to replacing the metric tensor $g_{\mu\nu}$ by the contravariant metric tensor density $g^{1/2} g^{\mu\nu}$ as the fundamental quantity describing the gravitational field.[15] The tensors $\phi_{\mu\nu}$ and $h_{\mu\nu}$ are related to each other like the tensors $\overset{1}{R}_{\mu\nu}$ and $\overset{1}{G}_{\mu\nu}$ (the superscript 1 denotes the first order part). Evidently $\phi_{\mu\mu} = -h_{\mu\mu}$ and $h_{\mu\nu} = \phi_{\mu\nu} - 1/2\, \delta_{\mu\nu}\, \phi_{\sigma\sigma}$.

We may record here the zero, first and second order terms, or the expan-

[15] Cf. R. C. Tolman, Relativity, Thermodynamics and Cosmology, Oxford (1934), Section 87, pp. 222-225.

sions themselves, of various useful field quantities:

$$g^{\mu\nu} = \delta_{\mu\nu} - h_{\mu\nu} + h_{\mu\alpha}\, h_{\alpha\nu} - \cdots \tag{5.7}$$

$$g^{1/2} = 1 + 1/2 h_{\mu\mu} + 1/8(h_{\mu\mu}\, h_{\nu\nu} - 2h_{\mu\nu}\, h_{\mu\nu}) + \cdots \tag{5.8}$$

$$[\mu\nu,\sigma] = 1/2(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}) \tag{5.9}$$

$$\overset{1}{R}_{\mu\nu\sigma\tau} = 1/2(h_{\mu\tau,\nu\sigma} - h_{\mu\sigma,\nu\tau} - h_{\nu\tau,\mu\sigma} + h_{\nu\sigma,\mu\tau}) \tag{5.10}$$

$$\begin{aligned} \overset{1}{G}_{\mu\nu} &= 1/2\left[h_{\mu\nu,\sigma\sigma} - h_{\mu\sigma,\nu\sigma} - h_{\nu\sigma,\mu\sigma} + h_{\sigma\sigma,\mu\nu} - \delta_{\mu\nu}(h_{\sigma\sigma,\tau\tau} - h_{\sigma\tau,\sigma\tau})\right] \\ &= 1/2(\phi_{\mu\nu,\alpha\alpha} - \phi_{\mu\alpha,\nu\alpha} - \phi_{\nu\alpha,\mu\alpha} + \delta_{\mu\nu}\phi_{\alpha\beta,\alpha\beta}) \end{aligned} \tag{5.11}$$

$$\begin{aligned} \overset{\circ}{\mathcal{L}}{}'_G &= -\frac{1}{4\beta}\left(2h_{\mu\nu,\mu}\, h_{\sigma\sigma,\nu} - h_{\mu\mu,\nu}\, h_{\sigma\sigma,\nu} - 2h_{\mu\sigma,\nu}\, h_{\mu\nu,\sigma} + h_{\mu\nu,\sigma}\, h_{\mu\nu,\sigma}\right) \\ &= -\frac{1}{4\beta}\left(\phi_{\mu\nu,\sigma}\phi_{\mu\nu,\sigma} - 2\phi_{\mu\nu,\sigma}\phi_{\mu\sigma,\nu} - 1/2\,\phi_{\mu\mu,\sigma}\phi_{\nu\nu,\sigma}\right) \end{aligned} \tag{5.12}$$

$$\begin{aligned} \overset{\circ}{T}_{G\mu\nu} = \frac{1}{4\beta}\Big[& \overset{2}{2\phi_{\alpha\beta,\mu}\phi_{\alpha\beta,\nu}} \overset{-1}{- \phi_{\alpha\alpha,\mu}\phi_{\beta\beta,\nu}} \overset{+4}{- 4\phi_{\alpha\beta,\mu}\phi_{\nu\alpha,\beta}} \\ & - \delta_{\mu\nu}(\overset{1}{\phi_{\alpha\beta,\tau}\phi_{\alpha\beta,\tau}} \overset{+2}{- 2\phi_{\alpha\beta,\tau}\phi_{\alpha\tau,\beta}} \overset{-\frac{1}{2}}{- 1/2\,\phi_{\alpha\alpha,\tau}\phi_{\beta\beta,\tau}})\Big] \end{aligned} \tag{5.13}$$

The notation $+\cdots$ or $-\cdots$ will always be understood to signify that the unwritten terms are of a higher order than the highest which occurs explicitly. The expansions (5.7) and (5.8) are obtained by simple application of the binomial theorem. The convergence of all expansions introduced, of course, depends on the smallness of $h_{\mu\nu}$.

The construction of corresponding expansions for quantities appearing in the spinor formalism is hindered by the fact that the Γ^μ and Ω_σ are not well defined functions of the $g_{\mu\nu}$. The simplest way to make Γ^μ and Ω_σ well defined is to construct them from the ordinary Dirac operators γ_μ by integrating (2.14) and (2.17) along a straight line path in the 10-dimensional space of the $g_{\mu\nu}$. One then obtains

$$\Gamma^\mu = \sqrt{g}^{\mu\nu}\gamma_\nu \quad , \quad \sqrt{g}^{\mu\nu} = \delta_{\mu\nu} - \tfrac{1}{2}h_{\mu\nu} + \frac{1\cdot 3}{2\cdot 4}\, h_{\mu\alpha}\, h_{\alpha\nu} - \cdots, \tag{5.14}$$

which is a simple binomial expansion. This prodedure, however, destroys the contravariant vector nature of Γ^μ. In the first place, since the γ_μ are to be kept constant, Γ^μ does not transform like a vector under Lorentz transformation. In the second place, Γ^μ, when defined by (5.14), does not transform properly under G-gauge transformations. The first problem can, of course, be solved by the usual method of placing the burden of Lorentz transformation upon the spinors in the theory. The second problem can likewise be solved by making the spinors share the burden of G-gauge transformation. According to (3.13), Γ^μ should obey the G-gauge transformation law

$$\underline{\delta}\Gamma^\mu = \Gamma^\mu{}_{,\nu}\delta\Lambda_\nu - \Gamma^\nu\delta\Lambda_{\mu,\nu} = (\sqrt{g}^{\mu\nu}{}_{,\sigma}\delta\Lambda_\sigma - \sqrt{g}^{\sigma\nu}\delta\Lambda_{\mu,\sigma})\gamma_\nu\,,$$

whereas, according to (5.14), it obeys the law

$$\begin{aligned}\delta\Gamma^\mu &= \delta\sqrt{g}^{\mu\nu}\gamma_\nu \\ &= (\sqrt{g}^{\mu\nu}{}_{,\sigma}\delta\Lambda_\sigma - \tfrac{1}{2}\delta\Lambda_{\mu,\nu} - \tfrac{1}{2}\delta\Lambda_{\nu,\mu} - \tfrac{1}{8}h_{\mu\alpha}\delta\Lambda_{\alpha,\nu} + \tfrac{3}{8}h_{\mu\alpha}\delta\Lambda_{\nu,\alpha} - \tfrac{1}{8}h_{\nu\alpha}\delta\Lambda_{\alpha,\mu} + \tfrac{3}{8}h_{\nu\alpha}\delta\Lambda_{\mu,\alpha} + \cdots)\gamma_\nu\end{aligned}$$

The difference can be made up by an infinitesimal spin space rotation $S = 1+\eta$ Explicitly,

$$\begin{aligned}[\eta,\gamma_\tau] &= \sqrt{g}_{\mu\tau}(\underline{\delta}\Gamma^\mu - \delta\Gamma^\mu) \\ &= -1/2\left[\delta\Lambda_{\tau,\nu} - \delta\Lambda_{\nu,\tau} + \tfrac{1}{4}(h_{\tau\alpha}\delta\Lambda_{\alpha,\nu} - h_{\nu\alpha}\delta\Lambda_{\alpha,\tau} + h_{\tau\alpha}\delta\Lambda_{\nu,\alpha} - h_{\nu\alpha}\delta\Lambda_{\tau,\alpha}) + \cdots\right]\gamma_\nu \qquad (5.15)\end{aligned}$$

where $\sqrt{g}_{\mu\nu}$ denotes the inverse to $\sqrt{g}^{\mu\nu}$. The solution of (5.15) is obtained with the use of (2.11) and we find that the G-gauge transformation law for a spinor has the form

$$\begin{aligned}\psi' &= \psi + \psi_{,\mu}\delta\Lambda_\mu - \eta\psi \\ &= \psi + \psi_{,\mu}\delta\Lambda_\mu - \tfrac{1}{16}\left[\delta\Lambda_{\mu,\nu} - \delta\Lambda_{\nu,\mu} + \tfrac{1}{4}(h_{\mu\alpha}\delta\Lambda_{\alpha,\nu} - h_{\nu\alpha}\delta\Lambda_{\alpha,\mu} + h_{\mu\alpha}\delta\Lambda_{\nu,\alpha} - h_{\nu\alpha}\delta\Lambda_{\mu,\alpha}) + \cdots\right][\gamma_\mu,\gamma_\nu]\psi\end{aligned} \qquad (5.16)$$

The construction of Ω_σ may be readily carried out by multiplying the identity $\Gamma^\mu{}_{,\sigma} + \left\{{\mu \atop \alpha\sigma}\right\}\Gamma^\alpha = -[\Omega_\sigma, \Gamma^\mu]$ by $\sqrt{g}_{\mu\nu}$. One obtains

$$\begin{aligned}[\Omega_\sigma,\gamma_\mu] &= -\sqrt{g}_{\mu\nu}\sqrt{g}^{\nu\tau}{}_{,\sigma}\gamma_\tau - \sqrt{g}_{\mu\nu}\left\{{\nu \atop \alpha\sigma}\right\}\sqrt{g}^{\alpha\tau}\gamma_\tau \\ &= -(\sqrt{g}^{\mu\nu}\, g_{\nu\alpha}\sqrt{g}^{\alpha\tau}{}_{,\sigma} + \sqrt{g}^{\mu\nu}[\alpha\sigma,\nu]\sqrt{g}^{\alpha\tau})\gamma_\tau \\ &= -(-\sqrt{g}^{\tau\nu}\, g_{\nu\alpha}\sqrt{g}^{\alpha\mu}{}_{,\sigma} + 1/2\sqrt{g}^{\mu\nu}(-g_{\alpha\nu,\sigma} + g_{\sigma\nu,\alpha} - g_{\alpha\sigma,\nu})\sqrt{g}^{\alpha\tau})\gamma_\tau \\ &= -1/2(\sqrt{g}^{\mu\alpha}\, g_{\alpha\beta}\sqrt{g}^{\beta\nu}{}_{,\sigma} - \sqrt{g}^{\nu\alpha}\, g_{\alpha\beta}\sqrt{g}^{\beta\mu}{}_{,\sigma} + \sqrt{g}^{\mu\alpha}\, g_{\sigma\alpha,\beta}\sqrt{g}^{\beta\nu} \\ &\qquad - \sqrt{g}^{\nu\alpha}\, g_{\sigma\alpha,\beta}\sqrt{g}^{\beta\mu})\gamma_\nu\end{aligned}$$

of which the solution with vanishing spur is given by

$$\Omega_\sigma = 1/16(\cdots\mu,\nu\cdots)\,[\gamma_\mu, \gamma_\nu]$$

$$= 1/16\Big[h_{\mu\sigma,\nu} - h_{\nu\sigma,\mu} + 1/4\,(2h_{\mu\alpha}\, h_{\nu\sigma,\alpha} - 2h_{\nu\alpha}\, h_{\mu\sigma,\alpha} - 2h_{\mu\alpha}\, h_{\alpha\sigma,\nu} + 2h_{\nu\alpha}\, h_{\alpha\sigma,\mu} - h_{\mu\alpha}\, h_{\alpha\nu,\sigma} + h_{\nu\alpha}\, h_{\alpha\mu,\sigma}) + \ldots\Big][\gamma_\mu, \gamma_\nu] \qquad (5.17)$$

where $(\cdots\mu,\nu\cdots)$ denotes the quantity inside the ~~brackets~~ parentheses in the preceding expression.

To first order, the G-gauge transformation law of the tensor $\phi_{\mu\nu}$ has the form

$$\phi'_{\mu\nu} = \phi_{\mu\nu} + \delta\Lambda_{\mu,\nu} + \delta\Lambda_{\nu,\mu} - \delta_{\mu\nu}\,\delta\Lambda_{\sigma,\sigma} + \ldots \qquad (5.18)$$

Taking the divergence of (5.18) one obtains $\phi'_{\mu\nu,\nu} = \phi_{\mu\nu,\nu} + \delta\Lambda_{\mu,\nu\nu} + \ldots$. To first order, G-guage transformation is a linear process, and a finite G-gauge transformation may be represented by a finite G-gauge parameter Λ_μ. By integrating in the 4-dimensional space of the Λ_μ from the origin to a value Λ_μ satisfying $\Lambda_{\mu,\nu\nu} = -\phi_{\mu\nu,\nu}$, one may carry out a G-gauge transformation which makes the divergence of $\phi_{\mu\nu}$ vanish to first order.

$$\phi_{\mu\nu,\nu} + \ldots = 0. \qquad (5.19)$$

Condition (5.19) is quite analagous to the Lorentz condition in electrodynamics, and when it is satisfied, the tensor $\overset{1}{G}_{\mu\nu}$ takes the particularly simple form

$$\overset{1}{G}_{\mu\nu} = 1/2\,\phi_{\mu\nu,\alpha\alpha} . \qquad (5.20)$$

Condition (5.19) may be used to evaluate the constant β . That is, condition (5.19) defines those <u>real space</u> coordinate systems in which we seem unconsciously to make our measurements; for the resulting equations describe what we actually observe to a high degree of accuracy. We need consider only static fields. For such fields the gravitational potentials $h_{\mu\nu}$ are of the order of magnitude of β rather than $\beta^{1/2}$. Hence we need consider only the first order interaction term $\overset{1}{\mathcal{L}}_M$ in the Lagrangian (5.3). Except for a possible ignorable divergence $\overset{1}{\mathcal{L}}_M$ is equal simply to $\frac{1}{2} h_{\mu\nu}\,\overset{0}{\Theta}_{M\,\mu\nu}$ or $1/2\,\phi_{\mu\nu}\,\overset{0}{\Theta}_{M\,\mu\nu} - 1/4\,\phi_{\mu\mu}\overset{0}{\Theta}_{M\,\nu\nu}$. With an impressed static external matter field, therefore, the appropriate Lagrangian function is

$$\mathcal{L} = \mathring{\mathcal{L}}'_G + 1/2\, \phi_{\mu\nu} \mathring{\Theta}_{M\mu\nu} - 1/4\, \phi_{\mu\mu} \mathring{\Theta}_{M\nu\nu}\,. \tag{5.21}$$

The field equations are the Einstein field equations to first order, which, with the G-gauge condition (5.19), take the form

$$\phi_{\mu\nu,\alpha\alpha} = -\beta\, \mathring{\Theta}_{M\mu\nu}\,. \tag{5.22}$$

The canonical energy momentum tensor for the Lagrangian (5.21) is

$$T_{\mu\nu} = \mathring{T}_{G\mu\nu} + \delta_{\mu\nu}\left(1/2\, \phi_{\alpha\beta} \mathring{\Theta}_{M\alpha\beta} - \tfrac{1}{4} \phi_{\alpha\alpha} \mathring{\Theta}_{M\beta\beta} \right) \tag{5.23}$$

and satisfies the equation $T_{\mu\nu,\nu} = 1/2\, \phi_{\alpha\beta} \mathring{\Theta}_{M\alpha\beta,\mu} - 1/4\, \phi_{\alpha\alpha} \mathring{\Theta}_{M\beta\beta,\mu}$ which shows explicitly the constancy in time of the total field energy when the matter distribution is static. Using (5.13), (5.19) and (5.22) we find for the total energy of the static field

$$E = -\int T_{44}\, d\boldsymbol{\tau} = \int \left[\frac{1}{4\beta} (\nabla\phi_{\alpha\beta} \cdot \nabla\phi_{\alpha\beta} - 2\phi_{\alpha i,j}\phi_{\alpha j,i} - \tfrac{1}{2} \nabla\phi_{\alpha\alpha} \cdot \nabla\phi_{\beta\beta}) - \tfrac{1}{2}\phi_{\alpha\beta}\mathring{\Theta}_{M\alpha\beta} + \tfrac{1}{4}\phi_{\alpha\alpha}\mathring{\Theta}_{M\beta\beta} \right] d\boldsymbol{\tau}$$

$$= \int \left(-1/4\, \phi_{\alpha\beta} \mathring{\Theta}_{M\alpha\beta} + \tfrac{1}{8} \phi_{\alpha\alpha} \mathring{\Theta}_{M\beta\beta} \right) d\boldsymbol{\tau}$$

Encircled in red ○ means use bold face type.

For point particles of masses m_i located at the points $\mathbf{r}_i$

$$\mathring{\Theta}_{M\mu\nu}(\mathbf{r}) = -\delta_{\mu 4}\delta_{\nu 4} \sum_i m_i c^2\, \delta(\mathbf{r} - \mathbf{r}_i) \tag{5.24}$$

which implies $$\phi_{\mu\nu}(\mathbf{r}) = -\delta_{\mu 4}\delta_{\nu 4} \frac{\beta c^2}{4\pi} \sum_i \frac{m_i}{|\mathbf{r} - \mathbf{r}_i|} \tag{5.25}$$

The total field energy becomes in this case

$$E = -1/8 \int \phi_{44} \mathring{\Theta}_{M44}\, d\boldsymbol{\tau} = -\frac{\beta c^4}{32\pi} \sum_{i,j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{5.26}$$

Evidently $$\beta = \frac{16\pi G}{c^4} \tag{5.27}$$

where G is the experimentally determined gravitational constant.

The constant β can also be determined from dynamical considerations. Returning momentarily to the covariant general coordinate notation of <u>real space</u>, we may write the current vector and stress tensor of a point particle having charge e and mass m moving in an impressed gravitational-electromagnetic field in the respective forms[16]

$$S^{\mu}(\xi) = e\int_{-\infty}^{\infty} \dot{\xi}^{\mu}(s)\, g^{-1/2}(\xi)\, \delta(\xi - \xi(s))\, ds, \qquad (5.28)$$

$$\Theta_M^{\mu\nu}(\xi) = mc^2\int_{-\infty}^{\infty} \frac{\dot{\xi}^{\mu}(s)\ \dot{\xi}^{\nu}(s)}{\left[-g_{\sigma\tau}(\xi(s))\,\dot{\xi}^{\sigma}(s)\,\dot{\xi}^{\tau}(s)\right]^{\frac{1}{2}}}\ g^{-1/2}(\xi)\,\delta(\xi - \xi(s))\,ds, \qquad (5.29)$$

where $\delta(\xi)$ is the 4-dimensional delta function,[17] $\xi^{\mu} = \xi^{\mu}(s)$ denotes the

[16] Forms (5.28) and (5.29) are essentially due to Dirac. Cf. Phys. Rev. 74 (1948), p. 818, equation (5).

[17] $g^{-1/2}(\xi)\ \delta(\xi - \xi(s))$ is evidently an invariant.

space-time path of the particle in terms of a parameter s whose values range from $-\infty$ to ∞ as the particle proceeds from the remote past to the remote future, and the dot denotes differentiation with respect to s. The correctness of forms (5.28) and (5.29) may be checked by evaluating them in a special Minkowski frame in flat space-time. The total charge is given by[18]

$$Q = \int_{\sigma} S^{\mu}(\xi)\, d\sigma_{\mu} = e\int_{\sigma}\left[\delta(\xi^2 - \xi^2(s))\,\delta(\xi^3 - \xi^3(s))\,\delta(\xi^0 - \xi^0(s))\,d\xi^2 d\xi^3 d\xi^0 + \cdots\right] = e$$

and is independent of the surface σ since the form (5.28), as may be readily shown, has vanishing divergence. The form (5.29), on the other hand, does not

[18] The surface element $d\sigma$ defined by three displacements $d_i\xi^{\mu}$ is given by $d\sigma = \left(3!\ g_{\mu_1\nu_1}\cdots\cdots g_{\mu_3\nu_3}\ d\sigma^{\mu_1\cdots\mu_3}\ d\sigma^{\nu_1\cdots\nu_3}\right)^{\frac{1}{2}} = \left|d_i\xi^{\mu}\ g_{\mu\nu}\ d_j\xi^{\nu}\right|^{\frac{1}{2}}$ where $d\sigma^{\mu_1\cdots\mu_3} = \frac{1}{3!}\,\epsilon_{i_1\cdots\cdots i_3}\ d_{i_1}\xi^{\mu_1}\cdots\cdots d_{i_3}\xi^{\mu_3}$. The surface element $d\sigma$ multiplied by its unit normal vector may be defined by $d\sigma_{\mu} = ig^{\frac{1}{2}}\underline{\delta}_{\mu\nu\sigma\tau}\,d\sigma^{\nu\sigma\tau}$ and when the displacements $d_i\xi^{\mu}$ are determined by the coordinate lattice, $d\sigma_{\mu}$ reduces to $(d\sigma_{\mu}) = \frac{1}{i}\, g^{\frac{1}{2}}\,(d\xi^2 d\xi^3 d\xi^4,\ d\xi^1 d\xi^3 d\xi^4,\ d\xi^1 d\xi^2 d\xi^4,\ d\xi^1 d\xi^2 d\xi^3)$. The volume element $d\omega$ defined by four displacements $d_{\nu}\xi^{\mu}$ is given by a similar expression: $d\omega = \frac{1}{i}\left(4!\ g_{\mu_1\nu_1}\cdots g_{\mu_4\nu_4}\ d\omega^{\mu_1\cdots\mu_4}\,d\omega^{\nu_1\cdots\nu_4}\right)^{\frac{1}{2}} = \frac{1}{i}\, g^{\frac{1}{2}}\left|d_{\nu}\xi^{\mu}\right|$

where $d\omega^{\mu_1\cdots\mu_4} = \frac{1}{4!}\epsilon_{\nu_1\cdots\nu_4} d_{\nu_1}\xi^{\mu_1}\cdots d_{\nu_4}\xi^{\mu_4}$. When the displacements are determined by the coordinate lattice, $d\omega$ reduces to $d\omega = g^{\frac{1}{2}} d\xi^1 d\xi^2 d\xi^3 d\xi^0$. These forms enable one, for example, to derive the Gauss theorem in general coordinates:

$$\int_\Omega \phi^\mu{}_{;\mu}\, d\omega = \int_\sigma \phi^\mu\, d\sigma_\mu$$

where σ is the surface of a region Ω, and ϕ^μ is a contravariant vector.

have identically vanishing divergence. One must, rather, impose a conservation law upon (5.29), whereupon the equations of motion of the particle become at once determined. If s is chosen to be the proper time $\bar{\tau}$ of the particle, then

$$\Theta_M{}^{\mu\nu}{}_{;\nu}(\xi) = mc\int_{-\infty}^{\infty} g^{-1/2}(\xi)\left[\ddot{\zeta}^\mu(\bar{\tau}) + \left\{{\mu \atop \nu\sigma}\right\}(\xi)\dot{\zeta}^\nu(\bar{\tau})\dot{\zeta}^\sigma(\bar{\tau})\right]\delta(\xi-\zeta(\bar{\tau}))\,d\bar{\tau} \qquad (5.30)$$

The imposition of the conservation law (3.25) leads to the equations of motion

$$\frac{d^2\xi^\mu}{d\bar{\tau}^2} + \left\{{\mu \atop \nu\sigma}\right\}\frac{d\xi^\nu}{d\bar{\tau}}\frac{d\xi^\sigma}{d\bar{\tau}} - \frac{e}{mc}F^\mu{}_\nu\frac{d\xi^\nu}{d\bar{\tau}} = 0\,. \qquad (5.31)$$

For vanishing charge these are the familiar geodesic equations.

In returning to the notation of the background space one must be careful to distinguish between the background proper time τ and the true proper time $\bar{\tau}$. Writing

$$\frac{d\bar{\tau}}{d\tau} = \left[1 - \frac{1}{c^2}h_{\mu\nu}(z(\tau))\,\dot{z}_\mu(\tau)\,\dot{z}_\nu(\tau)\right]^{1/2}, \qquad (5.32)$$

where $x_\mu = \xi^\mu = \zeta^\mu(\bar{\tau}) = z_\mu(\tau)$ denotes the particle path alternatively in real space or in the background space, one is led, in the case of an uncharged particle, from (5.31) to the ponderomotive equations

$$\frac{d^2x_\mu}{d\tau^2} + [\nu\sigma,\mu]\frac{dx_\nu}{d\tau}\frac{dx_\sigma}{d\tau} + \frac{1}{6c^2}(\nu\sigma\tau)\frac{dx_\mu}{d\tau}\frac{dx_\nu}{d\tau}\frac{dx_\sigma}{d\tau}\frac{dx_\tau}{d\tau} = 0 \qquad (5.33)$$

in which terms of order higher than the first have been neglected, and where

$$(\nu\sigma\tau) \equiv h_{\nu\sigma,\tau} + h_{\sigma\tau,\nu} + h_{\tau\nu,\sigma} \qquad (5.34)$$

Since $[\nu\sigma,\mu] + [\sigma\mu,\nu] + [\mu\nu,\sigma] = 1/2(\mu\nu\sigma)$, equations (5.33) are consistent with the necessary identity $U_\mu \frac{dU_\mu}{d\tau} = 0$, where $U_\mu = \frac{dx_\mu}{d\tau}$. If the gravitational field is static and of the form (5.25), then $h_{11} = h_{22} = h_{33} = -h_{44} = -1/2\phi_{44}$, $h_{\mu\nu} = 0$ for $\mu \neq \nu$, and equations (5.33) reduce to

$$\frac{dU_i}{d\tau} + \frac{c^2}{4}\left(\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right)\phi_{44,i} + \frac{1}{4}\left(\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right)\phi_{44,j}\, U_i\, U_j = 0 \qquad (5.35)$$

where v is the velocity of the particle in the background space. Equations (5.35) may be further reduced to the form

$$\frac{dv_i}{dt} + \frac{c^2}{4}\left(1+\frac{v^2}{c^2}\right)\phi_{44,i} = 0, \qquad (5.36)$$

which, except for the factor $1 + \frac{v^2}{c^2}$, has a very Newtonian appearance. For small velocities equation (5.36) describes the motion of a particle of mass m in a force field of potential $\frac{mc^2}{4}\phi_{44} = -\frac{\beta c^4}{16\pi}\sum_i \frac{m m_i}{|r - r_i|}$. Evidently, we are again led to the identification (5.27). It may be observed, incidentally, that equation (5.36) shows explicitly the well known double accelerating effect which a gravitational field has on a particle, e.g. a photon, which moves with the velocity v equal to c.

Two very pretty results may be obtained by observing that equations (5.33) may, to first order, be derived from a Lagrangian of the form

$$L = 1/2\, m\, U_\mu U_\mu + 3/4\, m h_{\mu\nu} U_\mu U_\nu + \frac{m}{4c^2} h_{\nu\sigma} U_\mu U_\mu U_\nu U_\sigma \qquad (5.37)$$

Conjugate momenta may be defined by $p_\mu = \frac{\partial L}{\partial U_\mu} = mU_\mu + m h_{\mu\nu} U_\nu + \frac{m}{2c^2} h_{\nu\sigma} U_\mu U_\nu U_\sigma$ and a Hamiltonian introduced: $H = U_\mu p_\mu - L = \frac{1}{2} m U_\mu U_\mu = -\frac{1}{2}mc^2$. To first order, the U_μ may be expressed in terms of the p_μ by

$$U_\mu = \frac{1}{m}\left(p_\mu - h_{\mu\nu} p_\nu - \frac{1}{2m^2c^2} h_{\nu\sigma} p_\mu p_\nu p_\sigma\right) \qquad (5.38)$$

and the Hamiltonian in the form

$$H = \frac{1}{2m}\left(p_\mu p_\mu - h_{\mu\nu} p_\mu p_\nu\right). \qquad (5.39)$$

This Hamiltonian may be applied in a quantum mechanical treatment of the interaction of the particle with the gravitational field. If we make the transition to quantum mechanics indicated by $p_\mu \to \frac{\hbar}{i}\frac{\partial}{\partial x_\mu}$, then the equation $H = -\frac{1}{2}mc^2$ is replaced by an operator equation $g^{\frac{1}{2}} H \psi = -\frac{1}{2}mc^2 g^{\frac{1}{2}}\psi$, in which the factor $g^{\frac{1}{2}}$ has been included in order to make a comparison with field equations derived from a Langrangian density. If we use the simplest method of symmetrizing the operators $h_{\mu\nu} p_\mu p_\nu$ and $h_{\mu\mu} p_\nu p_\nu$ so as to make them Hermitian, which is simply to write them in the forms $p_\mu h_{\mu\nu} p_\nu$ and $p_\nu h_{\mu\mu} p_\nu$ then this operator equation takes the explicit form

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_\mu^2} + \frac{1}{2}\frac{\partial}{\partial x_\mu} h_{\nu\nu}\frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x_\mu} h_{\mu\nu}\frac{\partial}{\partial x_\nu}\right)\psi = -\frac{mc^2}{2}\left(1 + \frac{1}{2}h_{\nu\nu}\right)\psi$$

or

$$\left(1+\tfrac{1}{2}h_{\alpha\alpha}\right)\psi_{,\mu\mu} + \tfrac{1}{2}h_{\nu\nu,\mu}\psi_{,\mu} - h_{\mu\nu}\psi_{,\mu\nu} - h_{\mu\nu,\mu}\psi_{,\nu} - K^2\left(1+\tfrac{1}{2}h_{\alpha\alpha}\right)\psi = 0. \tag{5.40}$$

But equation (5.40) is, to first order, simply the generalized Klein-Gordon equation (3.39) multiplied by $g^{\frac{1}{2}}$.

If the particle is a Dirac particle, a square root of the Hamiltonian must appear in the wave equation rather than the Hamiltonian itself. The simplest such square root is, to first order,

$$\sqrt{H} = \frac{1}{\sqrt{2m}}\gamma_\mu\left(p_\mu - \tfrac{1}{2}h_{\mu\nu}p_\nu\right) \tag{5.41}$$

After symmetrization of the operators in obvious fashion, the wave equation $g^{\frac{1}{2}}\sqrt{H}\,\psi = i\sqrt{\frac{m}{2}}\,c\,g^{\frac{1}{2}}\psi$ takes the form

$$-\frac{i\hbar}{\sqrt{2m}}\gamma_\mu\left(\frac{\partial}{\partial x_\mu} + \frac{1}{4}\frac{\partial}{\partial x_\mu}h_{\nu\nu} + \frac{1}{4}h_{\nu\nu}\frac{\partial}{\partial x_\mu} - \frac{1}{4}\frac{\partial}{\partial x_\nu}h_{\mu\nu} - \frac{1}{4}h_{\mu\nu}\frac{\partial}{\partial x_\nu}\right)\psi = i\sqrt{\frac{m}{2}}\,c\left(1+\tfrac{1}{2}h_{\nu\nu}\right)\psi$$

or

$$\gamma_\mu\left(\psi_{,\mu} + \tfrac{1}{2}h_{\alpha\alpha}\psi_{,\mu} + \tfrac{1}{4}h_{\alpha\alpha,\mu}\psi - \tfrac{1}{2}h_{\mu\nu}\psi_{,\nu} - \tfrac{1}{4}h_{\mu\nu,\nu}\psi\right) + K\left(1+\tfrac{1}{2}h_{\alpha\alpha}\right)\psi = 0. \tag{5.42}$$

And equation (5.42) is, to first order, simply the generalized Dirac equation (3.42) multiplied by $g^{\frac{1}{2}}$.

ABSTRACT

The Hamiltonian formulation of the "linearized" gravitational field equations is introduced. The longitudinal field components may be eliminated with the help of two auxiliary vector fields Λ_μ and Λ'_μ. The interaction Hamiltonian density for the interacting gravitational and scalar meson fields is constructed. When the longitudinal gravitational field is eliminated, a "Newtonian" term is introduced into the interaction. G-gauge transformations are discussed from the point of view of the interaction representation.

Perturbation approximation methods are applied in the calculation of the mesic stress induced in the vacuum by an impressed gravitational field. The induced stress is found to be logarithmically divergent and structure dependent, and hence not interpretable as a stress renormalization. The gravitational self-mass of the scalar meson is next calculated, and is found to be ambiguous, depending, to second order, on what gravitational tensor is chosen with which to make power expansions. By appropriate choice of this tensor the self-mass can, however, be made to assume the unique finite value $\delta m = -(G/2\pi\hbar c)m^3$. The self-energy operator, itself, remains quadratically divergent.

The same calculations are also carried out for the interacting gravitational and electromagnetic fields. In order that the supplementary Lorentz condition always be maintained, it is shown that every G-gauge transformation must be accompanied by a transformation in the electromagnetic gauge.

The vacuum induced electromagnetic stress is found to be also logarithmically divergent and structure dependent. Owing to the tracelessness of the stress tensor, on the other hand, the self-energy operator is in this case unambiguous and is found, moreover, to be <u>identically zero</u>.

ON THE APPLICATION OF QUANTUM PERTURBATION THEORY

TO GRAVITATIONAL INTERACTIONS.

II. INTERACTION REPRESENTATION, VACUUM INDUCED STRESS, SELF-ENERGIES OF MESON AND PHOTON.

Bryce Seligman DeWitt*
Harvard University, Cambridge, Massachusetts.

1. The "Free" Gravitational Field.

In ~~a~~ the previous paper[1] the Einstein-Mie theory of gravitational inter-

[1] ~~Phys. Rev.~~ (1950). This paper will hereafter be denoted by (I).

actions has been discussed, the notion of "G-gauge transformation" in a "background space" has been introduced, and, finally an approximation method for calculating specific physical effects has been outlined. One assumes that it is possible to introduce a coordinate system into the space-time manifold, with respect to which the metric tensor takes the form $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, where the $h_{\mu\nu}$ are small compared to unity. One then expands the Lagrangian of the combined gravitational and matter fields in ascending powers of the $h_{\mu\nu}$.

zero

$$\mathcal{L}' = \overset{0\prime}{\mathcal{L}}_G + \mathcal{L}_G'^{\,SELF} + \overset{0}{\mathcal{L}}_M + \overset{1}{\mathcal{L}}_M + \overset{2}{\mathcal{L}}_M + \cdots \tag{1.1}$$

(Here, all the higher order terms in the expansion of $\mathcal{L}_G'$ have been lumped together under the single designation $\mathcal{L}_G'^{\,self}$.) This method of approximation differs from the well-known "weak field" approximation only in the fact that $h_{\mu\nu}$ is not, itself, expanded in ascending powers of some small parameter.

The quantity $\mathcal{L}_G'^{\,self}$ contains all those terms which describe the interaction of the gravitational field with itself and hence, in the quantized theory, all the terms which are ambiguous by reason of containing non-commuting factors. In this paper we shall consider only problems in which it is never necessary to examine the explicit form of $\mathcal{L}_G'^{\,self}$.

* NRC Postdoctoral Fellow, now at the Institute for Advanced Study, Princeton, N.J.

(a) Since the work will be carried out in an interaction representation, according to the Schwinger-Tomonaga invariant prescription, it will be necessary to have available the mathematical properties of the quantized "free" or "linearized" gravitational field which is described by the Lagrangian $\overset{\circ}{\mathcal{L}}{}'_G$. (It may be objected that the field described by $\overset{\circ}{\mathcal{L}}{}'_G$ corresponds to the physically real gravitational field only in a very limited sense. But such an objection is no more nor less valid than a corresponding objection to the whole philosophy of perturbation theory and the interaction representation.) The Lagrangian $\overset{\circ}{\mathcal{L}}{}'_G$, in the form

$$\overset{\circ}{\mathcal{L}}{}'_G = -\frac{1}{4\beta}\left(\phi_{\mu\nu,\sigma}\phi_{\mu\nu,\sigma} - 2\phi_{\mu\nu,\sigma}\phi_{\mu\sigma,\nu} - \tfrac{1}{2}\phi_{\mu\mu,\sigma}\phi_{\nu\nu,\sigma}\right) \qquad (1.2)$$

(see (I, 5.12)), where $\phi_{\mu\nu} = h_{\mu\nu} - 1/2\,\delta_{\mu\nu} h_{\sigma\sigma}$ and β is the gravitational coupling constant, cannot be used directly in the construction of a Hamiltonian formalism since the conjugate momenta do not turn out to be algebraically independent. The Hamiltonian formulation is, however, readily carried out if we replace $\overset{\circ}{\mathcal{L}}{}'_G$ by

$$\hat{\mathcal{L}}_G = -\frac{1}{4\beta}\left(\phi_{\mu\nu,\sigma}\phi_{\mu\nu,\sigma} - \tfrac{1}{2}\phi_{\mu\mu,\sigma}\phi_{\nu\nu,\sigma}\right) \qquad (1.3)$$

which simply omits the middle term of (1.2), and impose the supplementary condition

$$\phi_{\mu\nu,\nu}\,\Phi = 0 \qquad (1.4)$$

on the state vector of the quantized field (see (I, 5.19)). Condition (1.4) is analagous to the Lorentz condition of quantum electrodynamics. The field equations generated by $\hat{\mathcal{L}}_G$ are

$$\phi_{\mu\nu,\sigma\sigma} - \tfrac{1}{2}\delta_{\mu\nu}\phi_{\alpha\alpha,\sigma\sigma} = 0 \qquad (1.5)$$

which, together with their contracted form, imply

$$\Box^2\phi_{\mu\nu} = 0. \qquad (1.6)$$

In virtue of condition (1.4), the field equations which $\overset{\circ}{\mathcal{L}}{}'_G$ generates can be written as eigenvalue equations

$$\overset{1}{G}_{\mu\nu}\,\Phi = 0. \tag{1.7}$$

The tensor $\overset{1}{G}_{\mu\nu}$ is defined in (I), equation (5.11).

Conjugate momenta $\Pi_{\mu\nu}$ may be introduced by the definition

$$\Pi_{\mu\nu} = \frac{1}{c} n_\sigma \frac{\partial \hat{\mathcal{L}}_G}{\partial \phi_{\mu\nu,\sigma}} = -\frac{1}{4\beta c} n_\sigma (2\phi_{\mu\nu,\sigma} - \delta_{\mu\nu}\phi_{\alpha\alpha,\sigma}) \tag{1.8}$$

where n_μ is an arbitrary time-like unit vector[2] in the background space.

[2] Unit time-like vectors will always be given a negative time-like orientation in correspondence to the customary practice of giving such negative orientation to the unit normals to a space-like surface. A proper Lorentz transformation can always then be made which brings n_μ into the form $(n_\mu)=(0,0,0,-i)$.

It is to be noted, however, that in virtue of the symmetry of $\phi_{\mu\nu}$, $\Pi_{\mu\nu}$ and $\Pi_{\nu\mu}$ are not independent quantities. The true momentum conjugate to $\phi_{\mu\nu}$ is $2\Pi_{\mu\nu}$ for $\mu\neq\nu$ and $\Pi_{\mu\nu}$ for $\mu=\nu$. If $\sigma_n(x)$ is a flat space-like surface perpendicular to n_μ and passing through x, and f(x) is an arbitrary function of x, then the following commutation relations may immediately be written.

$$\int_{\sigma_n(x)} f(x')\,[\phi_{\mu\nu}(x), \Pi_{\sigma\tau}(x')]\,d\sigma' = \frac{i\hbar}{2}(\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma}) f(x) \qquad \text{for } \mu\neq\nu$$

$$\int_{\sigma_n(x)} f(x')\,[\phi_{\mu\mu}(x), \Pi_{\sigma\sigma}(x')]\,d\sigma' = i\hbar\,\delta_{\mu\sigma} f(x) \qquad \text{(no summation over } \mu \text{ and } \sigma).$$

Using the inverse to (1.8), namely $n_\sigma\phi_{\mu\nu,\sigma} = -2\beta c\left(\Pi_{\mu\nu}-\frac{1}{2}\delta_{\mu\nu}\Pi_{\alpha\alpha}\right)$ these equations may be combined into the single equation

$$\int_\sigma f(x')\,[\phi_{\mu\nu}(x), \phi_{\sigma\tau,\alpha}(x')]\,d\sigma'_\alpha = -i\hbar\beta c\,(\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma}-\delta_{\mu\nu}\delta_{\sigma\tau}) f(x) \tag{1.9}$$

where, in virtue of the Lorentz invariance of the theory, the flat surface $\sigma_n(x)$ is replaced by an arbitrary surface σ passing through x. Equations (1.6) and (1.9) together imply

$$[\phi_{\mu\nu}(x), \phi_{\sigma\tau}(x')] = i\hbar\beta c\,(\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma}-\delta_{\mu\nu}\delta_{\sigma\tau})\,D(x-x') \text{ [3]} \tag{1.10}$$

[3] Since $[\phi_{\mu\nu,\nu}(x), \phi_{\sigma\tau,\tau}(x')] = 0$, the supplementary condition (1.4) is consistent with these relations.

where D(x) is the invariant δ-function for zero mass. The tensor $h_{\mu\nu}$ satisfies identical commutation relations.

$$[h_{\mu\nu}(x), h_{\sigma\tau}(x')] = i\hbar\beta c(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\tau})D(x-x'). \quad (1.11)$$

As we have noted in (I), G-gauge transformation is, to first order, a linear process. Under a G-gauge transformation the linearized field $\phi_{\mu\nu}$ is transformed according to

$$\phi'_{\mu\nu} = \phi_{\mu\nu} + \Lambda_{\mu,\nu} + \Lambda_{\nu,\mu} - \delta_{\mu\nu}\Lambda_{\sigma,\sigma} \quad (1.12)$$

where Λ_μ is the now finite G-gauge parameter. Equations (1.4) and (1.6) remain invariant under G-gauge transformations for which the parameter Λ_μ satisfies the wave equation $\Box^2\Lambda_\mu = 0$. This invariance may be used to carry out a covariant elimination of the longitudinal components of the linearized gravitational field by a procedure precisely analagous to that employed by Schwinger[4]

[4] Phys. Rev. 74, 1439 (1948), section 3.

to eliminate the longitudinal electromagnetic field components.

One introduces two vector fields Λ_μ and Λ'_μ (in analogy with Schwinger's scalar fields Λ and Λ') which satisfy the following equations.

$$\Box^2\Lambda_\mu = 0, \qquad \Box^2\Lambda'_\mu = 0 \quad (1.13)$$

$$\phi_{\mu\mu} = 4n_\mu n_\nu(\Lambda_{\mu,\nu} - \Lambda'_{\mu,\nu}) + 2\Lambda'_{\mu,\mu} \quad (1.14)$$

$$n_\nu\phi_{\mu\nu} = -n_\nu(\Lambda_{\mu,\nu} + \Lambda_{\nu,\mu}) + 2n_\mu n_\nu n_\sigma(\Lambda_{\nu,\sigma} - \Lambda'_{\nu,\sigma}) + n_\mu\Lambda'_{\nu,\nu} \quad (1.15)$$

$$\phi_{\mu\nu,\nu} = n_\mu n_\nu(\Lambda_{\sigma,\sigma\nu} - \Lambda'_{\sigma,\sigma\nu}) + n_\nu n_\sigma(\Lambda_{\mu,\nu\sigma} - \Lambda'_{\mu,\nu\sigma}) + n_\nu n_\sigma(\Lambda_{\nu,\mu\sigma} - \Lambda'_{\nu,\mu\sigma}). \quad (1.16)$$

In virtue of equation (1.16) the supplementary condition (1.4) may evidently be replaced by the condition

$$(\Lambda_\mu - \Lambda'_\mu)\,\Phi = 0 \; . \quad (1.17)$$

If, now, a special G-gauge is introduced, by means of the G-gauge transformation

$$\begin{aligned}\phi_{\mu\nu} = \phi_{\mu\nu} &+ \Lambda'_{\mu,\nu} + \Lambda'_{\nu,\mu} - \delta_{\mu\nu}\Lambda'_{\sigma,\sigma} - n_\mu n_\sigma(\Lambda_{\nu,\sigma} - \Lambda'_{\nu,\sigma}) \\ &- n_\nu n_\sigma(\Lambda_{\mu,\sigma} - \Lambda'_{\mu,\sigma}) - n_\nu n_\sigma(\Lambda_{\sigma,\mu} - \Lambda'_{\sigma,\mu}) - n_\mu n_\sigma(\Lambda_{\sigma,\nu} - \Lambda'_{\sigma,\nu}),\end{aligned} \quad (1.18)$$

the new field tensor $\phi_{\mu\nu}$ will, invirtue of equations (1.14 - 1.16), satisfy the relations

$$\phi_{\mu\mu} = 0, \quad n_\nu \phi_{\mu\nu} = 0, \quad \phi_{\mu\nu,\nu} = 0 \tag{1.19}$$

which indicate that the transformation has completely eliminated the longitudinal components.

In order to obtain the commutation relations satisfied by $\Lambda_\mu, \Lambda'_\mu$ and $\phi_{\mu\nu}$ it is necessary to introduce two new functions, $\overset{1}{\mathcal{D}}(x)$ and $\overset{2}{\mathcal{D}}(x)$ which satisfy the equations[5]

$$\Box^2 \overset{1}{\mathcal{D}} = 0, \qquad \Box^2 \overset{2}{\mathcal{D}} = 0$$

$$D = n_\mu n_\nu \overset{1}{\mathcal{D}}_{,\mu\nu} = \partial_\mu \partial_\mu \overset{1}{\mathcal{D}}, \qquad \overset{1}{\mathcal{D}} = n_\mu n_\nu \overset{2}{\mathcal{D}}_{,\mu\nu} = \partial_\mu \partial_\mu \overset{2}{\mathcal{D}} \tag{1.20}$$

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[5] $\overset{1}{\mathcal{D}}$ (x) is Schwinger's function $\mathcal{D}$ (x). If one wishes to carry out a complete program of covariant elimination of the longitudinal components of all higher-spin massless fields (including half-integral-spin fields), it is necessary to introduce a hierarchy of functions $\overset{n}{\mathcal{D}}$ satisfying

$$\Box^2 \overset{n}{\mathcal{D}} = 0, \quad \overset{n-1}{\mathcal{D}} = n_\mu n_\nu \overset{n}{\mathcal{D}}_{,\mu\nu} = \partial_\mu \partial_\mu \overset{n}{\mathcal{D}}, \quad n = 0, 1, 2, \cdots \quad \overset{0}{\mathcal{D}} = D .$$

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where the operator ∂_μ is defined by

$$\partial_\mu \equiv \frac{\partial}{\partial x_\mu} - n_\mu \frac{\partial}{\partial n}, \qquad \frac{\partial}{\partial n} \equiv - n_\mu \frac{\partial}{\partial x_\mu} . \tag{1.21}$$

The explicit construction of $\overset{2}{\mathcal{D}}$ for a special coordinate system is given in the appendix (A.17). By direct substitution into equations (1.14 - 1.18) the validity of the following commutation relations may be verified.

$$[\Lambda_\mu(x), \Lambda_\nu(x')] = - [\Lambda'_\mu(x), \Lambda'_\nu(x')]$$
$$= i\hbar\beta c \left(\delta_{\mu\nu} \overset{1}{\mathcal{D}} - \tfrac{1}{2} n_\mu n_\sigma \overset{2}{\mathcal{D}}_{,\nu\sigma} - \tfrac{1}{2} n_\nu n_\sigma \overset{2}{\mathcal{D}}_{,\mu\sigma} - \tfrac{1}{4} \overset{2}{\mathcal{D}}_{,\mu\nu}\right)(x - x') \tag{1.22}$$

$$[\Lambda_\mu(x), \Lambda'_\nu(x')] = 0 \tag{1.23}$$

$$[\Lambda_\mu(x), \phi_{\nu\sigma}(x')] = i\hbar\beta c \left(-\delta_{\mu\nu} n_\sigma n_\alpha \overset{1}{\mathcal{D}}_{,\alpha} - \delta_{\mu\sigma} n_\nu n_\alpha \overset{1}{\mathcal{D}}_{,\alpha} + n_\nu n_\sigma \overset{1}{\mathcal{D}}_{,\mu}\right)(x - x') \tag{1.24}$$

$$[\Lambda'_\mu(x), \phi_{\nu\sigma}(x')] = - i\hbar\beta c \Big[\delta_{\mu\nu}(\overset{1}{\mathcal{D}}_{,\sigma} + n_\sigma n_\tau \overset{1}{\mathcal{D}}_{,\tau}) + \delta_{\mu\sigma}(\overset{1}{\mathcal{D}}_{,\nu} + n_\nu n_\tau \overset{1}{\mathcal{D}}_{,\tau}) - (\tfrac{1}{2}\delta_{\nu\sigma} + n_\nu n_\sigma)\overset{1}{\mathcal{D}}_{,\mu}$$
$$- \tfrac{1}{2} \overset{2}{\mathcal{D}}_{,\mu\nu\sigma} - \tfrac{1}{2} n_\nu n_\tau \overset{2}{\mathcal{D}}_{,\tau\sigma\mu} - \tfrac{1}{2} n_\sigma n_\tau \overset{2}{\mathcal{D}}_{,\tau\nu\mu} - n_\mu n_\tau \overset{2}{\mathcal{D}}_{,\tau\nu\sigma}\Big](x - x') \tag{1.25}$$

$$[\Lambda_\mu(x), \phi_{\nu\sigma}(x')] = [\Lambda'_\mu(x), \phi_{\nu\sigma}(x')] = 0 . \tag{1.26}$$

The commutator of $\phi_{\mu\nu}(x)$ with $\phi_{\sigma\tau}(x')$ can be obtained with the aid of relations

(1.22 - 1.26) and turns out to be a rather lengthy expression. If, however, one introduces the operator

$$d_{\mu\nu} \equiv (\delta_{\mu\nu} + n_\mu n_\nu)\partial_\alpha\partial_\alpha - \partial_\mu\partial_\nu \tag{1.27}$$

this expression can be greatly reduced. A straightforward but tedious calculation shows that

$$[\phi_{\mu\nu}(x), \phi_{\sigma\tau}(x')] = i\hbar\beta c(d_{\mu\sigma}d_{\nu\tau} + d_{\mu\tau}d_{\nu\sigma} - d_{\mu\nu}d_{\sigma\tau})\overset{2}{D}(x-x'). \tag{1.28}$$

The operator $d_{\mu\nu}$ satisfies the relations

$$d_{\mu\mu} = 2\partial_\mu\partial_\mu, \quad n_\mu d_{\mu\nu} = 0, \quad d_{\mu\nu,\nu} = 0, \quad d_{\mu\sigma}d_{\sigma\nu} = d_{\mu\nu}\partial_\alpha\partial_\alpha, \tag{1.29}$$

which may be used to show the consistency of (1.28) with equations (1.19). Equations (1.23) and (1.26) show that $\Lambda_\mu, \Lambda'_\mu$ and $\phi_{\mu\nu}$ are dynamically independent. Of the 10 independent components of $\varphi_{\mu\nu}$, Λ_μ and Λ'_μ together account for 8, leaving 2 to be accounted for by $\phi_{\mu\nu}$. These two "transverse" components of $\phi_{\mu\nu}$ correspond to the two possible polarization states of a linearized gravitational wave.

The canonical energy-momentum tensor constructed from $\mathring{\mathcal{L}}'_G$ may be shown to differ from that constructed from $\hat{\mathcal{L}}_G$ in amount given by

$$\mathring{T}_{G\mu\nu} - \hat{T}_{G\mu\nu} = \frac{1}{2\beta}(\delta_{\mu\nu}\varphi_{\alpha\beta}\varphi_{\alpha\sigma,\beta} - \delta_{\mu\sigma}\varphi_{\alpha\beta}\varphi_{\alpha\nu,\beta} + \varphi_{\alpha\sigma}\varphi_{\alpha\nu,\mu} - \varphi_{\alpha\nu}\varphi_{\alpha\sigma,\mu})_{,\sigma}$$

$$+ \frac{1}{2\beta}(\varphi_{\alpha\beta,\beta\mu}\varphi_{\alpha\nu} - \varphi_{\alpha\beta,\beta}\varphi_{\alpha\nu,\mu} - \delta_{\mu\nu}\varphi_{\alpha\beta}\varphi_{\alpha\sigma,\sigma\beta})$$

The supplementary condition (1.4) evidently allows us to write the expectation value equations

$$\langle \mathring{T}_{G\mu\nu,\nu} \rangle = \langle \mathring{\Theta}_{G\mu\nu,\nu} \rangle = 0 \tag{1.30}$$

$$\langle \mathring{P}_{G\mu} \rangle = \frac{1}{c}\int_\sigma \langle \mathring{T}_{G\mu\nu} \rangle d\sigma_\nu = \frac{1}{c}\int_\sigma \langle \hat{T}_{G\mu\nu} \rangle d\sigma_\nu = \langle \hat{P}_{G\mu} \rangle. \tag{1.31}$$

If $\varphi_{\mu\nu}$ is replaced by $\phi_{\mu\nu}$ in the defining equations for $\hat{T}_{G\mu\nu}$ and $\mathring{T}_{G\mu\nu}$ then $\mathring{P}_{G\mu} = \hat{P}_{G\mu}$. The significance of this 4-vector as a displacement operator is still retained provided all field dependent quantities are expressed in terms of $\phi_{\mu\nu}$ alone. For, invirtue of the G-gauge invariance of $\mathring{P}_{G\mu}$,

$\overset{\circ}{P}_{G\mu}(\phi) = \overset{\circ}{P}_{G\mu}(\phi^*) = \hat{P}_{G\mu}(\phi^*)$ where $\phi^*_{\mu\nu} = \phi_{\mu\nu} - \Lambda'_{\mu,\nu} - \Lambda'_{\nu,\mu} + \delta_{\mu\nu}\Lambda'_{\sigma,\sigma}$ and

$$[\overset{\circ}{P}_{G\mu}(\phi), \phi_{\alpha\beta}] = [\hat{P}_{G\mu}(\phi^*), \phi_{\alpha\beta}] = [\hat{P}_{G\mu}(\phi), \phi_{\alpha\beta}] + \frac{1}{c}\int_\sigma f[\Lambda_\mu(x') - \Lambda'_\mu(x')]\, d\sigma'$$

where $f[\Lambda_\mu - \Lambda'_\mu]$ is some function of $\Lambda_\mu - \Lambda'_\mu$ and its first and second derivatives. Since $\Lambda_\mu - \Lambda'_\mu$ commutes with itself at any two points of space-time and is everywhere dynamically independent of $\phi_{\mu\nu}$, $f[\Lambda_\mu - \Lambda'_\mu]$ must evidently vanish, and we are left with

$$[\overset{\circ}{P}_{G\mu}(\phi), \phi_{\alpha\beta}] = [\hat{P}_{G\mu}(\phi), \phi_{\alpha\beta}] = i\hbar\, \phi_{\alpha\beta,\mu} \,. \tag{1.32}$$

When all field quantities are expressed in terms of $\phi_{\mu\nu}$ alone, the complete set of basic vectors in state-vector space may be restricted to include only eigenvectors of the operator $\Lambda_\mu - \Lambda'_\mu$ corresponding to the eigenvalue zero, and $\Lambda_\mu - \Lambda'_\mu$ may be dropped from the theory.

The vacuum state vector Ψ_0 of the "free" gravitational system is defined by

$$\phi^{(+)}_{\mu\nu}\, \Psi_0 = 0, \tag{1.33}$$

where (+) denotes the positive frequency part. Following Schwinger[6] we may ex-

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[6] Phys. Rev. 75, 651 (1949), section 3.

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tend this definition by adding the conditions $\Lambda^{(+)}_\mu \Psi_0 = \Lambda'^{(+)}_\mu \Psi_0 = 0$, thus yielding

$$\phi^{*(+)}_{\mu\nu}\, \Psi_0 = 0 \tag{1.34}$$

In the case of the "free" gravitational system nothing of a physical nature is added by adopting the definition (1.34). However, a complete demonstration of the equivalence of definitions (1.33) and (1.34) in the interaction representation, in the cases in which other fields are interacting with the gravitational field and the gravitational field is interacting with itself, is an extremely complicated and tedious procedure, involving the complexities of a full discussion of all first and second order interaction terms, and has not been carried out by the author. Presumably such a demonstration is nevertheless possible, in analogy with the similar demonstration in electrodynamics, and in subsequent self-energy calculations the validity of definition (1.34) will be assumed.

If one makes the attempt to demonstrate the equivalence of (1.33) and (1.34) in the case of interacting fields, one finds it necessary to compare the values of the derivatives of the functions $\overset{1}{\mathcal{D}}{}^{(1)}$ and $\overset{2}{\mathcal{D}}{}^{(1)}$ at the origin with the value of the function $D^{(1)}$ at the origin.[7] These values are infinite, but

[7] $D^{(1)} = i(D^{(+)} - D^{(-)})$, etc.

not ambiguous. Because of invariance requirements, $\overset{1}{\mathcal{D}}{}^{(1)}_{,\mu\nu}(0)$, for example, must have the form $A\,\delta_{\mu\nu} + B\,n_\mu n_\nu$. In virtue of equations (1.20), A and B are determined to have the values $\frac{1}{3}D^{(1)}(0)$ and $\frac{4}{3}D^{(1)}(0)$ respectively.

$$\overset{1}{\mathcal{D}}{}^{(1)}_{,\mu\nu}(0) = \left(\tfrac{1}{3}\delta_{\mu\nu} + \tfrac{4}{3}n_\mu n_\nu\right)D^{(1)}(0)\,. \tag{1.35}$$

Similarly,

$$\overset{2}{\mathcal{D}}{}^{(1)}_{,\mu\nu\sigma\tau}(0) = \Big[\tfrac{1}{15}(\delta_{\mu\nu}\delta_{\sigma\tau}+\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma}) + \tfrac{2}{5}(\delta_{\mu\nu}n_\sigma n_\tau + \delta_{\sigma\tau}n_\mu n_\nu + \delta_{\mu\sigma}n_\nu n_\tau + \delta_{\nu\tau}n_\mu n_\sigma + \delta_{\mu\tau}n_\nu n_\sigma + \delta_{\nu\sigma}n_\mu n_\tau) + \tfrac{16}{5}n_\mu n_\nu n_\sigma n_\tau\Big]D^{(1)}(0)\,. \tag{1.36}$$

From the anticommutator vacuum expectation value equations

$$\langle\{\phi_{\mu\nu}(x), \phi_{\sigma\tau}(x')\}\rangle_0 \doteq \hbar\beta c(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\tau})D^{(1)}(x-x'), \tag{1.37}$$

$$\langle\{\phi_{\mu\nu}(x), \phi_{\sigma\tau}(x')\}\rangle_0 = \hbar\beta c(d_{\mu\sigma}d_{\nu\tau} + d_{\mu\tau}d_{\nu\sigma} - d_{\mu\nu}d_{\sigma\tau})\overset{2}{\mathcal{D}}{}^{(1)}(x-x'), \tag{1.38}$$

one may, for example, make the comparison of

$$\langle\{\phi_{\mu\nu}, \phi_{\sigma\tau}\}\rangle_0 = \hbar\beta c(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\tau})D^{(1)}(0) \tag{1.39}$$

with

$$\langle\{\phi_{\mu\nu}, \phi_{\sigma\tau}\}\rangle_0 = \hbar\beta c\Big[\tfrac{2}{5}(\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma}) - \tfrac{4}{15}\delta_{\mu\nu}\delta_{\sigma\tau} + \tfrac{2}{5}(\delta_{\mu\sigma}n_\nu n_\tau + \delta_{\nu\tau}n_\mu n_\sigma + \delta_{\mu\tau}n_\nu n_\sigma + \delta_{\nu\sigma}n_\mu n_\tau) - \tfrac{4}{15}(\delta_{\mu\nu}n_\sigma n_\tau + \delta_{\sigma\tau}n_\mu n_\nu) + \tfrac{8}{15}n_\mu n_\nu n_\sigma n_\tau\Big]D^{(1)}(0)\,. \tag{1.40}$$

Equation (1.40) is readily verified to be consistent with (1.19).

2. The Scalar Meson Field.

The simplest example to which the methods of a quantum perturbation theory of gravitation can be applied is that of the interacting gravitational and scalar meson fields. Before choosing the appropriate Lagrangian density function to describe this interaction let us first examine the energy-momentum tensor of the free meson field.

$$\overset{\circ}{\Theta}_{M\mu\nu} = \overset{\circ}{T}_{M\mu\nu} = \tfrac{1}{2}\{\phi_{,\mu}, \phi_{,\nu}\} - \tfrac{1}{2}\delta_{\mu\nu}(\phi_{,\sigma}\phi_{,\sigma} + \kappa^2\phi^2) \tag{2.1}$$

The vacuum expectation of this tensor is

$$\langle \overset{\circ}{\Theta}_{M\mu\nu} \rangle_0 = -\frac{\hbar c}{2}\Delta^{(1)}_{,\mu\nu}(0) = -\frac{1}{8}\hbar c\,\kappa^2\delta_{\mu\nu}\Delta^{(1)}(0). \tag{2.2}$$

Since we must obviously not couple this physically meaningless, quadratically divergent, vacuum stress to the gravitational field we must add $\frac{1}{8}\hbar c\,\kappa^2\delta_{\mu\nu}\Delta^{(1)}(0)$ to the definition of $\overset{\circ}{\Theta}_{M\mu\nu}$. This is equivalent to adding a term $\frac{1}{8}\hbar c\,\kappa^2\Delta^{(1)}(0)\,g^{\frac{1}{2}}$ to the Lagrangian density $\mathcal{L}_M$ of (I, 3.38).

The interacting gravitational and meson fields are therefore, in the quantized theory, correctly described by a total Lagrangian of the form

$$\mathcal{L} = \hat{\mathcal{L}}_G + \hat{\mathcal{L}}_G^{SELF} - \tfrac{1}{2}g^{\frac{1}{2}}(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \kappa^2\phi^2 - \tfrac{1}{4}\hbar c\,\kappa^2\Delta^{(1)}(0)) \tag{2.3}$$

together with a supplementary condition (cf. I, 5.19)

$$(\varphi_{\mu\nu,\nu} + \cdots)\,\Phi = 0, \tag{2.4}$$

which, to the first order at least, is the same as (1.4). The bold faced type is here used to distinguish the Heisenberg representation, and the gravitational self-interaction term $\mathcal{L}_G'^{\,\text{self}}$ of (1.1) is replaced by $\hat{\mathcal{L}}_G^{\text{self}}$ in accordance with the possible necessity of making minor alterations in this term so that the supplementary condition (2.4) will be compatible with the full field equations. We shall never make an explicit examination of the form of $\hat{\mathcal{L}}_G^{\text{self}}$ or of the higher order terms in condition (2.4), but they must presumably be of such a nature that the Einstein field equations (I, 3.9) may be written as eigenvalue equations.

$$\left(G^{\mu\nu} + \frac{\beta}{2} \Theta_M{}^{\mu\nu} \right) \Phi = 0 . \tag{2.5}$$

With the use of the expansions (I, 5.7, 5.8) the total meson Lagrangian density function may be expanded in the form

$$\mathcal{L}_M = -\tfrac{1}{2} g^{\frac{1}{2}} \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \kappa^2 \phi^2 - \tfrac{1}{4} \hbar c \kappa^2 \Delta^{(1)}(0) \right) = \overset{0}{\mathcal{L}}_M + \overset{1}{\mathcal{L}}_M + \overset{2}{\mathcal{L}}_M + \cdots$$

where $$\overset{1}{\mathcal{L}}_M = \tfrac{1}{2} h_{\mu\nu} \left[\phi_{,\mu} \phi_{,\nu} - \tfrac{1}{2} \delta_{\mu\nu} \left(\phi_{,\sigma} \phi_{,\sigma} + \kappa^2 \phi^2 - \tfrac{1}{4} \hbar c \kappa^2 \Delta^{(1)}(0) \right) \right] = \tfrac{1}{2} h_{\mu\nu} \overset{0}{\Theta}_{M\,\mu\nu} \tag{2.6}$$

and $$\begin{aligned} \overset{2}{\mathcal{L}}_M = -\tfrac{1}{2} \Big(& \varphi_{\mu\alpha} \varphi_{\alpha\nu} \phi_{,\mu} \phi_{,\nu} - \tfrac{1}{2} \varphi_{\alpha\alpha} \varphi_{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \tfrac{1}{8} \varphi_{\alpha\alpha} \varphi_{\beta\beta} \phi_{,\mu} \phi_{,\mu} - \tfrac{1}{4} \varphi_{\alpha\beta} \varphi_{\alpha\beta} \phi_{,\mu} \phi_{,\mu} \\ & + \frac{\kappa^2}{8} \varphi_{\alpha\alpha} \varphi_{\beta\beta} \phi^2 - \frac{\kappa^2}{4} \varphi_{\alpha\beta} \varphi_{\alpha\beta} \phi^2 - \frac{\hbar c \kappa^2}{32} \Delta^{(1)}(0) \varphi_{\alpha\alpha} \varphi_{\beta\beta} + \frac{\hbar c \kappa^2}{16} \Delta^{(1)}(0) \varphi_{\alpha\beta} \varphi_{\alpha\beta} \Big). \end{aligned} \tag{2.7}$$

Introducing a unit time-like vector n_μ, conjugate momenta may be defined:

$$\pi = \frac{1}{c} n_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = \frac{1}{c} n_\mu \left(-\phi_{,\mu} + \varphi_{\mu\nu} \phi_{,\nu} + \cdots \right) = \frac{1}{c} \left(\frac{\partial \phi}{\partial n} + n_\mu n_\nu \varphi_{\mu\nu} \frac{\partial \phi}{\partial n} + \frac{1}{c} n_\mu \varphi_{\mu\nu} \partial_\nu \phi + \cdots \right), \tag{2.8}$$

$$\pi_{\mu\nu} = \frac{1}{c} n_\sigma \frac{\partial \mathcal{L}}{\partial \varphi_{\mu\nu,\sigma}} = \frac{1}{4\beta c} \left(2 \frac{\partial \varphi_{\mu\nu}}{\partial n} - \delta_{\mu\nu} \frac{\partial \varphi_{\alpha\alpha}}{\partial n} \right) + \frac{1}{c} n_\sigma \frac{\partial \hat{\mathcal{L}}_G^{\text{SELF}}}{\partial \varphi_{\mu\nu,\sigma}} \quad \text{(cf. (1.8))} \tag{2.9}$$

and a Hamiltonian density constructed:

$$\begin{aligned} \mathcal{H} = n_\mu n_\nu T_{\mu\nu} &= n_\mu n_\nu \left(-\varphi_{\alpha\beta,\mu} \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha\beta,\nu}} - \phi_{,\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} + \delta_{\mu\nu} \mathcal{L} \right) \\ &= c \frac{\partial \varphi_{\alpha\beta}}{\partial n} \pi_{\alpha\beta} + c \frac{\partial \phi}{\partial n} \pi - \mathcal{L} . \end{aligned} \tag{2.10}$$

With the use of the inverses to (2.8) and (2.9), namely

$$\frac{\partial \phi}{\partial n} = c\pi - c n_\mu n_\nu \varphi_{\mu\nu} \pi - n_\mu \varphi_{\mu\nu} \partial_\nu \phi + \cdots , \tag{2.11}$$

$$\frac{\partial \varphi_{\mu\nu}}{\partial n} = 2\beta c \left(\pi_{\mu\nu} - \tfrac{1}{2} \delta_{\mu\nu} \pi_{\alpha\alpha} \right) - 2\beta n_\sigma \left(\frac{\partial \hat{\mathcal{L}}_G^{\text{SELF}}}{\partial \varphi_{\mu\nu,\sigma}} - \frac{1}{2} \delta_{\mu\nu} \frac{\partial \hat{\mathcal{L}}_G^{\text{SELF}}}{\partial \varphi_{\alpha\alpha,\sigma}} \right), \tag{2.12}$$

$\mathcal{H}$ may be obtained, by straightforward algebra, in the form

$$\begin{aligned} \mathcal{H} = \overset{0}{\mathcal{H}}_G + \overset{0}{\mathcal{H}}_M + \mathcal{H}_G^{\text{SELF}} & - \tfrac{1}{2} \varphi_{\mu\nu} \left(\partial_\mu \phi + c n_\mu \pi \right) \left(\partial_\nu \phi + c n_\nu \pi \right) + \frac{\kappa^2}{4} h_{\alpha\alpha} \phi^2 \\ & - \frac{\hbar c \kappa^2}{16} \Delta^{(1)}(0) h_{\alpha\alpha} + \frac{c^2}{2} \left(n_\mu n_\nu \varphi_{\mu\nu} \pi + \frac{1}{c} n_\mu \varphi_{\mu\nu} \partial_\nu \phi \right)^2 - \overset{2}{\mathcal{L}}_M + \cdots \end{aligned} \tag{2.13}$$

where $\mathcal{H}_G{}^{\text{self}}$ is the operator which describes the self-interactions of the gravitational field and $\mathring{\mathcal{H}}_G$ and $\mathring{\mathcal{H}}_M$ are the Hamiltonian densities for the free fields.

$$\mathring{\mathcal{H}}_M = \frac{1}{2}\left(c^2\Pi^2 + \partial_\mu\phi\,\partial_\mu\phi + \kappa^2\phi^2 - \frac{1}{4}\hbar c\kappa^2\Delta^{(1)}(0)\right) \tag{2.14}$$

$$\mathring{\mathcal{H}}_G = \frac{1}{4\beta}\left(4\beta^2c^2\,\Pi_{\mu\nu}\Pi_{\mu\nu} - 2\beta^2c^2\,\Pi_{\mu\mu}^2 + \partial_\sigma\phi_{\mu\nu}\partial_\sigma\phi_{\mu\nu} - \frac{1}{2}\partial_\sigma\phi_{\mu\mu}\partial_\sigma\phi_{\nu\nu}\right). \tag{2.15}$$

Returning to the use of ordinary type for denoting variables in the interaction representation, we may write

$$\Pi = \frac{1}{c}\frac{\partial\phi}{\partial n} \quad . \tag{2.16}$$

The transition from (2.13) to the interaction representation is immediate. The interaction Hamiltonian density becomes

$$\mathcal{H}_{INT}[x,\sigma] = \mathcal{H}_G^{SELF}[x,\sigma] + \overset{1}{\mathcal{H}}_{INT}(x) + \overset{2}{\mathcal{H}}_{INT}[x,\sigma] + \cdots \tag{2.17}$$

where $\quad \overset{1}{\mathcal{H}}_{INT} = -\frac{1}{2}h_{\mu\nu}\,\mathring{\Theta}_{M\mu\nu}$ (2.18)

and $\quad \overset{2}{\mathcal{H}}_{INT}[x,\sigma] = \left[\frac{1}{2}(n_\mu\phi_{\mu\nu}\phi_{,\nu})^2 - \mathcal{L}_M\right][x,\sigma]$ (2.19)

the fixed unit vector n_μ having now been replaced by the variable unit vector $n_\mu[x,\sigma]$ normal to a space-like surface σ at the point x.

Before making use of the interaction Hamiltonian density $\mathcal{H}_{INT}[x,\sigma]$ in the calculations of a particular physical problem we must make sure that it satisfies the integrability condition[8]

$$\left[\mathcal{H}_{INT}[x,\sigma], \mathcal{H}_{INT}[x',\sigma]\right] - i\hbar c\left(\frac{\delta}{\delta\sigma(x)}\mathcal{H}_{INT}[x',\sigma] - \frac{\delta}{\delta\sigma(x')}\mathcal{H}_{INT}[x,\sigma]\right) = 0 \tag{2.20}$$

for two points x and x' lying in σ and hence separated by a space-like interval. We cannot, of course, check this condition for that part of the interaction

[8] S. Kanesawa and S. Tomonaga, Prog. Theor. Phys. 3, 101 (1948).

Hamiltonian density which refers to the self-interactions of the gravitational field since we do not know the explicit form of $\mathcal{H}_G^{SELF}[x,\sigma]$. But since we are only interested in the interactions between the meson and gravitational fields, we need check it only for that part of $\mathcal{H}_{INT}[x,\sigma]$ which refers to such interactions. Furthermore, since all calculations will be carried out only up to the second order in $\beta^{\frac{1}{2}}$ we need check it only to that degree of accuracy. Using the commutation relation

$$[\phi(x), \phi(x')] = i\hbar c\, \Delta(x-x') \tag{2.21}$$

and (A.11) of the Appendix, one finds, after considerable reduction,

$$\begin{aligned} [\overset{1}{\mathcal{H}}_{INT}(x), \overset{1}{\mathcal{H}}_{INT}(x')] &= i\hbar c\left(\frac{\delta}{\delta\sigma(x)}\overset{2}{\mathcal{H}}_{INT}[x',\sigma] - \frac{\delta}{\delta\sigma(x')}\overset{2}{\mathcal{H}}_{INT}[x,\sigma]\right) \\ &= -i\hbar c\, \Delta_{,\mu\nu}(x-x')\, \phi_{,\alpha}\, \phi_{,\beta}\, \phi_{\mu\alpha}\, \phi_{\nu\beta} \end{aligned} \tag{2.22}$$

where the field variables in the final expression may be evaluated either at the point x or the point x'. The integrability condition is therefore satisfied up to the required second order.

In the interaction representation the field variables satisfy the "free-field" equations $(\Box^2-\kappa^2)\phi = 0$, $\Box^2\phi_{\mu\nu} = 0$. If transition is made back to the Heisenberg representation, it will be found that the interaction Hamiltonian density $\mathcal{H}_{INT}[x,\sigma]$ regenerates the original field equations. We shall verify this only for the meson field equations, and only up to the first order. Using equations (A.12) and (A.13) of the Appendix, we find, after calculating the indicated commutators, carrying out the integrations with the help of (A.7) and (A.8) of the Appendix, and simplifying the resulting expressions,

$$U[\sigma]\,\phi_{,\mu}(x)\,U^*[\sigma] = (\phi_{,\mu} - n_\mu n_\alpha \phi_{\alpha\beta}\, \phi_{,\beta})[x,\sigma] + \cdots \tag{2.23}$$

$$\begin{aligned} U[\sigma]\,\phi_{,\mu\nu}(x)\,U^*[\sigma] = {}& \phi_{,\mu\nu}(x) \\ & - [(k\, n_\alpha n_\mu n_\nu + n_\alpha \partial_\mu n_\nu + n_\mu \partial_\alpha n_\nu + n_\nu \partial_\alpha n_\mu)\phi_{\alpha\beta}\,\phi_{,\beta} \\ & + n_\mu n_\nu (\phi_{\alpha\beta}\,\phi_{,\beta})_{,\alpha} + n_\mu n_\alpha (\phi_{\alpha\beta}\,\phi_{,\beta})_{,\nu} + n_\nu n_\alpha (\phi_{\alpha\beta}\,\phi_{,\beta})_{,\mu} \\ & + 2 n_\mu n_\nu n_\alpha n_\gamma (\phi_{\alpha\beta}\,\phi_{,\beta})_{,\gamma} + \tfrac{1}{2}\kappa^2 n_\mu n_\nu \phi_{\alpha\alpha}\,\phi][x,\sigma] \\ & + \cdots\cdots \end{aligned} \tag{2.24}$$

where $U[\sigma]$ is the unitary operator which connects the Heisenberg and interaction representations and satisfies the equation

$$i\hbar c\,\frac{\delta U[\sigma]}{\delta\sigma(x)} = \mathcal{H}_{INT}[x,\sigma]\,U[\sigma]\,. \tag{2.25}$$

By contracting equation (2.24) and using $(\square^2 - \kappa^2)\phi = 0$, we get

$$U[\sigma](\phi_{,\mu\mu} - \kappa^2\phi)(x)\, U^*[\sigma] = \left[(\phi_{\alpha\beta}\phi_{,\beta})_{,\alpha} - \tfrac{1}{2}\kappa^2\phi_{\alpha\alpha}\phi\right](x) + \cdots \tag{2.26}$$

Multiplying this equation on the left and right by $U^*[\sigma]$ and $U[\sigma]$ respectively, and replacing $\phi_{\alpha\beta}$ by $h_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta}h_{\gamma\gamma}$, one obtains, to first order, the generalized Klein-Gordon equation of (I, 5.40).

The corresponding equations for the tensor $\phi_{\mu\nu}$ are complicated, naturally enough, by the presence of the operator $\mathcal{H}_G^{SELF}[x,\sigma]$ in the interaction Hamiltonian density. One finds

$$U[\sigma]\phi_{\mu\nu,\sigma}(x)\, U^*[\sigma] = \phi_{\mu\nu,\sigma}(x) + \frac{1}{i\hbar c}\int_\sigma [\phi_{\mu\nu}(x'), \mathcal{H}_G^{SELF}[x,\sigma]]\, d\sigma'_\sigma \tag{2.27}$$

$$\begin{aligned} U[\sigma]\phi_{\mu\nu,\sigma\tau}(x)\, U^*[\sigma] = {} & (\phi_{\mu\nu,\sigma\tau} + \beta n_\sigma n_\tau \mathring{\Theta}_{m\,\mu\nu})[x,\sigma] + \cdots \\ & + \frac{1}{i\hbar c}\int_\sigma [\phi_{\mu\nu,\sigma}(x'), \mathcal{H}_G^{SELF}[x,\sigma]]\, d\sigma'_\tau \\ & + \frac{1}{i\hbar c} U[\sigma]\left(\frac{\delta}{\delta\sigma(x)}\int_\sigma\int_\sigma U^*[\sigma][\phi_{\mu\nu}(x''), \mathcal{H}_G^{SELF}[x',\sigma]]U[\sigma]\, d\sigma''_\sigma\, d\sigma'_\tau\right) U^*[\sigma]. \end{aligned} \tag{2.28}$$

Equation (2.27) may be used to express the supplementary gauge condition (2.4) in the interaction representation. Letting $\Psi[\sigma] = U[\sigma]\Phi$ denote the interaction state vector, we have

$$\phi_{\mu\nu,\nu}(x)\Psi[\sigma] = -\left[U[\sigma]\{\cdots\mu\cdots\}(x) + \frac{1}{i\hbar c}\int_\sigma [\phi_{\mu\nu}(x'), \mathcal{H}_G^{SELF}[x,\sigma]]\, d\sigma'_\nu\, U[\sigma]\right]\Phi \tag{2.29}$$

where $\{\cdots\mu\cdots\}(x)$ denotes the unwritten higher order terms in (2.4). Presumably the form of $\mathcal{H}_G^{SELF}[x,\sigma]$ must be of such a nature that the integral in (2.29) exactly cancels these unwritten terms, leaving us with the exact equation

$$\phi_{\mu\nu,\nu}(x)\Psi[\sigma] = 0. \tag{2.30}$$

Assuming the validity of equation (2.30) (upon which assumption, indeed, the full validity of the interaction representation rests) we may generalize it, with the help of equation (2.28), to the case in which x is not restricted to lie on the surface σ. Using the identity $f(x) \equiv \int_\sigma [D_{,\mu}(x'-x)f(x') - D(x'-x)f_{,\mu}(x')]\, d\sigma'_\mu$ which holds for any function f which satisfies the free wave equation $\square^2 f = 0$, and the fact that

$$\{\cdots\mu\cdots\}_{,\sigma}(x') = \frac{\delta}{\delta\sigma(x')}\int_\sigma \{\cdots\mu\cdots\}(x'')\,d\sigma_\sigma'' = -\frac{1}{i\hbar c}\frac{\delta}{\delta\sigma(x')}\int_\sigma\int_\sigma U^*[\sigma]\left[\phi_{\mu\nu}(x'''), \mathcal{H}_G^{SELF}[x'',\sigma]\right]U[\sigma]\,d\sigma_\nu'''\,d\sigma_\sigma''$$

we may write

$$\phi_{\mu\nu,\nu}(x)\,\Psi[\sigma] = \int_\sigma \left[D_{,\sigma}(x'-x)\,\phi_{\mu\nu,\nu}(x') - D(x'-x)\,\phi_{\mu\nu,\nu\sigma}(x')\right]d\sigma_\sigma'\;\Psi[\sigma]$$

$$= -\int_\sigma D(x'-x)\Big[U[\sigma]\,\phi_{\mu\nu,\nu\sigma}(x')\,U^*[\sigma] - \beta\, n_\nu' n_\sigma'\,\mathring{\Theta}_{M\mu\nu}(x') - \cdots$$

$$-\frac{1}{i\hbar c}\int_\sigma \left[\phi_{\mu\nu,\nu}(x''), \mathcal{H}_G^{SELF}[x',\sigma]\right]d\sigma_\sigma'' + U[\sigma]\{\cdots\mu\cdots\}_{,\sigma}(x')\,U^*[\sigma]\Big]\,d\sigma_\sigma'\;\Psi[\sigma]\,.$$

In virtue of (2.4), the first term inside the brackets in the above integral cancels the last term, and we are left with

$$\left[\phi_{\mu\nu,\nu}(x) - \int_\sigma D(x-x')\left(\beta\,\mathring{\Theta}_{M\mu\nu}(x') - \frac{1}{i\hbar c}\int_\sigma\left[\phi_{\mu\nu,\sigma}(x'), \mathcal{H}_G^{SELF}[x'',\sigma]\right]d\sigma_\sigma'' + \cdots\right)d\sigma_\nu'\right]\Psi[\sigma] = 0 \qquad (2.31)$$

This generalized supplementary condition may be compared with the supplementary condition introduced by Schwinger in the case of electrodynamics (Phys. Rev. 74, 1439 (1948), equation 2.32). The stress here takes the place of the current. An essentially new feature, however, appears in the fact that the gravitational field itself, as well as the matter field, contributes to the stress. Presumably the term involving $\mathcal{H}_G^{SELF}[x'',\sigma]$ in (2.31) is actually nothing more than the gravitational stress $\Theta_{G\mu\nu}$, or something closely akin to it.

In the interaction representation, a change of G-gauge becomes the linear process indicated by (1.12). When transition is made back to the Heisenberg representation the non-linear terms in the G-gauge transformation law, such as occur in (I, 5.2), are generated by the self interaction term $\mathcal{H}_G^{SELF}[x,\sigma]$ in the interaction Hamiltonian density. In the interaction representation, the meson field variable ϕ remains unchanged under a G-gauge transformation, but in the Heisenberg representation it transforms according to

$$\phi' = \phi + \phi_{,\mu}\,\delta\Lambda_\mu \qquad (2.32)$$

The unitary operator which connects the Heisenberg and interaction representations must evidently suffer change under a G-gauge transformation. We may write its transformation law in the form

$$U'[\sigma] = \left(1 + i\,S[\sigma]\right) U[\sigma] \qquad (2.33)$$

where $S[\sigma]$ is an infinitesimal Hermitian operator and a functional of the G-gauge parameter $\delta\Lambda_\mu$, which satisfied the equations

$$i\left[\phi(x), S[\sigma]\right] = U[\sigma]\,\phi_{,\mu}(x)\,U^*[\sigma]\,\delta\Lambda_\mu(x) \qquad (2.34)$$

$$i\left[h_{\mu\nu}(x), S[\sigma]\right] = U[\sigma]\left(h_{\mu\nu,\sigma}\delta\Lambda_\sigma + h_{\mu\sigma}\delta\Lambda_{\sigma,\nu} + h_{\nu\sigma}\delta\Lambda_{\sigma,\mu}\right)(x)\,U^*[\sigma]\,. \qquad (2.35)$$

In the above equations $\delta\Lambda_\mu$ is a c-number and hence dynamically independent of the fields ϕ and $\phi_{\mu\nu}$. This being the case, it is easy to obtain an explicit expression for $S[\sigma]$ to the first order. The interaction Hamiltonian density suffers a G-gauge transformation which is given simply by

$$\delta\mathcal{H}_{INT}[x,\sigma] = -\delta\Lambda_{\mu,\nu}(x)\,\mathring{\Theta}_{M\mu\nu}(x) + \delta\mathcal{H}_G^{SELF}[x,\sigma] + \cdots \qquad (2.36)$$

The transformed "equation of motion," $i\hbar c\,\frac{\delta U'[\sigma]}{\delta\sigma(x)} = \left(\mathcal{H}_{INT} + \delta\mathcal{H}_{INT}\right)[x,\sigma]\,U'[\sigma]$, leads to the condition

$$-\hbar c\,\frac{\delta S[\sigma]}{\delta\sigma(x)} + i\left[S[\sigma], \mathcal{H}_{INT}[x,\sigma]\right] = \delta\mathcal{H}_{INT}[x,\sigma], \qquad (2.37)$$

which has the following solution, to first order.[9]

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[9] The G-gauge parameter $\delta\Lambda_\mu$, in addition to being infinitesimal, is regarded as a first order quantity. Also, a finite, integrated G-gauge parameter is a first order quantity.

$$S[\sigma] = \frac{1}{\hbar c}\int_\sigma \mathring{\Theta}_{M\mu\nu}\,\delta\Lambda_\mu\,d\sigma_\nu - \frac{1}{\hbar c}\int_{-\infty}^{\sigma} \delta\mathcal{H}_G^{SELF}[x',\sigma']\,d\omega' + \cdots \qquad (2.38)$$

This solution is readily shown to satisfy (2.34) to first order.

The operator defined by equation (2.38) may be used in effecting a covariant elimination of the longitudinal components of the gravitational field. One simply integrates in the 4-dimensional G-gauge-parameter-space so as to obtain a finite G-gauge parameter, which is then chosen to be the vector Λ'_μ of equations (1.13 - 1.16). The G-gauge parameter is accordingly no longer a c-number. The unitary operator describing the G-gauge transformation takes the form

$$u[\sigma] = e^{iS[\sigma]} \qquad (2.39)$$

where $S[\sigma] = \frac{1}{\hbar c}\int_\sigma \mathring{\Theta}_{M\mu\nu}\,\Lambda'_\mu\,d\sigma_\nu - \frac{1}{\hbar c}\int_{-\infty}^{\sigma} \Delta\mathcal{H}_G^{SELF}[x',\sigma']\,d\omega' + \cdots$

The transformed interaction Hamiltonian density is given by

$$\begin{aligned}\mathcal{H}'_{INT}[x,\sigma] &= \mathcal{H}_{INT}[x,\sigma] + \left[U[\sigma], \mathcal{H}_{INT}[x,\sigma]\right] + i\hbar c \frac{\delta U[\sigma]}{\delta\sigma(x)} U^*[\sigma] \\ &= \mathcal{H}_{INT}[x,\sigma] + i\left[S[\sigma], \mathcal{H}_{INT}[x,\sigma]\right] - \hbar c \frac{\delta S[\sigma]}{\delta\sigma(x)} - \frac{i\hbar c}{2}\left[S[\sigma], \frac{\delta S[\sigma]}{\delta\sigma(x)}\right] + \cdots \qquad (2.39)\end{aligned}$$

It is of interest to carry out an explicit calculation of this expression, including one second order term, namely that term which is the gravitational analogue for the matter field of the static Coulomb term of electrodynamics. This term may conveniently be called the <u>Newtonian term.</u> Using the commutator equations (1.22) and (1.25), we find

$$\mathcal{H}'_{INT}[x,\sigma] = \mathcal{H}_{INT}[x,\sigma] - \Lambda'_{\mu,\nu}(x)\overset{\circ}{\Theta}_{M\mu\nu}(x) + \Delta\mathcal{H}_G^{SELF}[x,\sigma] + \mathcal{H}_N[x,\sigma] + \cdots ,$$

where the Newtonian term, $\mathcal{H}_N[x,\sigma]$, is given by

$$\begin{aligned}\mathcal{H}_N[x,\sigma] = -\frac{\beta}{4}\int_\sigma \Big[&\delta_{\mu\sigma}\big(\overset{1}{\mathcal{D}}_{,\tau} + 2n_\tau n_\alpha \overset{1}{\mathcal{D}}_{,\alpha}\big) - \tfrac{1}{2}\delta_{\sigma\tau}\big(\overset{1}{\mathcal{D}}_{,\mu} + 2n_\mu n_\alpha \overset{1}{\mathcal{D}}_{,\alpha}\big) \qquad (2.40) \\ &- n_\sigma n_\tau \overset{1}{\mathcal{D}}_{,\mu} - \tfrac{1}{4}\overset{2}{\mathcal{D}}_{,\mu\sigma\tau} - \tfrac{1}{2}n_\sigma n_\alpha \overset{2}{\mathcal{D}}_{,\alpha\tau\mu} \\ &- \tfrac{1}{2}n_\mu n_\alpha \overset{2}{\mathcal{D}}_{,\alpha\sigma\tau}\Big](x-x')\left\{\overset{\circ}{\Theta}_{M\mu\nu}(x'), \overset{\circ}{\Theta}_{M\sigma\tau}(x)\right\} d\sigma'_\nu\end{aligned}$$

(There must also, of course, exist a Coulomb-analogue term for the gravitational field itself, describing its "static"[10] self-interaction.)

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[10] Since the gravitational field is a radiation type field, the term "static" cannot apply to it in the strict sense.

Before examining the form of the Newtonian term in a special coordinate system we should note how the supplementary condition (2.31) is altered by the G-gauge transformation (2.39). First of all, in virtue of equation (1.16), condition (2.31) may be rewritten in the form

$$\begin{aligned}\Big[&n_\mu n_\nu\big(\Lambda_{\sigma,\sigma\nu}(x) - \Lambda'_{\sigma,\sigma\nu}(x)\big) + n_\nu n_\sigma\big(\Lambda_{\mu,\nu\sigma}(x) - \Lambda'_{\mu,\nu\sigma}(x)\big) + n_\nu n_\sigma\big(\Lambda_{\nu,\mu\sigma}(x) - \Lambda'_{\nu,\mu\sigma}(x)\big) \\ &- \int_\sigma D(x-x')\Big(\beta\overset{\bullet}{\Theta}_{M\mu\nu}(x') - \frac{1}{i\hbar c}\int_\sigma \big[\phi_{\mu\nu,\sigma}(x'), \mathcal{H}_G^{SELF}[x'',\sigma]\big] d\sigma''_\sigma + \cdots\Big) d\sigma'_\nu\Big]\Psi[\sigma] = 0. \qquad (2.41)\end{aligned}$$

Taking the divergence of (2.41) and also multiplying it by n_μ , one obtains equations which, together with (2.41), imply the following.

$$\Big[\Lambda_\mu(x) - \Lambda'_\mu(x) - \int_\sigma \Big(\delta_{\mu\sigma}\overset{1}{D} - \tfrac{1}{2}n_\mu n_\alpha \overset{2}{D}_{,\alpha\sigma} - \tfrac{1}{2}n_\sigma n_\alpha \overset{2}{D}_{,\alpha\mu} - \tfrac{1}{4}\overset{2}{D}_{,\mu\sigma}\Big)(x-x') \tag{2.42}$$

$$\times\Big(\beta\overset{\circ}{\Theta}_{M\sigma\tau}(x') - \frac{1}{i\hbar c}\int_\sigma [\phi_{\sigma\tau,\alpha}(x'), \mathcal{H}_G^{SELF}[x'',\sigma]]\,d\sigma''_\alpha + \cdots\Big)d\sigma'_\tau\Big]\Psi[\sigma] = 0$$

Multiplying this equation on the left by $U[\sigma]$, one obtains, on using the commutation relations (1.22),

$$\Big[\Lambda_\mu(x) - \Lambda'_\mu(x) + \frac{i}{\hbar c}\int_{-\infty}^{\sigma}[\Lambda_\mu(x) - \Lambda'_\mu(x), \Delta\mathcal{H}_G^{SELF}[x',\sigma']]\,d\omega' \tag{2.43}$$

$$-\frac{i}{\hbar c}\int_\sigma\int_\sigma\Big(\delta_{\mu\sigma}\overset{1}{D} - \tfrac{1}{2}n_\mu n_\alpha\overset{2}{D}_{,\alpha\sigma} - \tfrac{1}{2}n_\sigma n_\alpha\overset{2}{D}_{,\alpha\mu} - \tfrac{1}{4}\overset{2}{D}_{,\mu\sigma}\Big)(x-x')[\phi_{\sigma\tau,\alpha}(x'), \mathcal{H}_G^{SELF}[x'',\sigma]]\,d\sigma''_\alpha\, d\sigma'_\tau + \cdots\Big]\Psi'[\sigma] = 0.$$

The terms involving the matter field are now seen to have disappeared from the supplementary condition. Presumably the gravitational terms also, in reality, cancel each other, leaving us simply with

$$[\Lambda_\mu(x) - \Lambda'_\mu(x)]\,\Psi'[\sigma] = 0. \tag{2.44}$$

By dropping the operator $\Lambda_\mu - \Lambda'_\mu$ from the theory, as was done in section 1, the elimination of the longitudinal components becomes complete. The transformed interaction Hamiltonian density takes the form

$$\mathcal{H}'_{INT}[x,\sigma] = -\tfrac{1}{2}\phi_{\mu\nu}(x)\overset{\circ}{\Theta}_{M\mu\nu}(x) + \mathcal{H}_G^{SELF}[x,\sigma] + \mathcal{H}_N[x,\sigma] + \cdots \tag{2.45}$$

where $\phi_{\mu\nu}$ is the transverse field of section 1 and replaces $\phi_{\mu\nu}$ in the expression for $\mathcal{H}_G^{SELF}[x,\sigma]$.

In a coordinate system in which n_μ takes the form (0,0,0,-i) the total static gravitational interaction energy in the matter field may be readily calculated with the aid of equations (A.18) of the Appendix. After considerable reduction, the expression for this energy is found to have the form

$$\int\mathcal{H}_N[\mathbf{r},t,\sigma_n]\,d\mathbf{r}$$

$$= -\frac{G}{c^4}\iint\Big(\frac{1}{4}\frac{\{U',U\}}{|\mathbf{r}-\mathbf{r}'|} + \frac{5}{8}\frac{\{U',\Theta_{ii}\}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{c^2}\frac{\{S'_i,S_i\}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{8}\frac{(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}:\{U',\boldsymbol{\Theta}\}\Big)d\mathbf{r}\,d\mathbf{r}' \tag{2.46}$$

where G is the gravitational constant, U the matter energy density, $\mathbf{S}$ the energy flux vector, and $\boldsymbol{\Theta}$ the stress dyadic. Equation (2.46) is quite general and applies to all matter fields (including the electromagnetic field) as well as to the scalar meson field. If it is applied to the example of point particles at rest, then all terms but the first vanish in the integrand, and the first term yields the familiar Newtonian potential (I, 5.26). Strictly speakin only this first term can properly be called Newtonian.

3. Mesic Stress Induced in the Vacuum.

The simplest purely quantum problem, to which the foregoing formalism can be applied, is that of calculating the mesic stress which is induced in the vacuum as a result of the interaction of an impressed external gravitational field with the vacuum fluctuations of the meson field. The phenomenon is quite analogous to the vacuum polarization effect of electrodynamics.

The meson field, initially in its vacuum state, is perturbed by the establishment of an externally generated gravitational field. The G-gauge group will be restricted by requiring the gravitational potentials $h_{\mu\nu}$ to vanish prior to the creation of the gravitational field, as well as after its final removal. Since the gravitational field is externally impressed, the potentials $h_{\mu\nu}$ are effectively c-numbers in this calculation, and the purely gravitational term $\mathcal{H}_G^{SELF}[x,\sigma]$ in the interaction Hamiltonian density must be omitted. Moreover, to find the leading term in the formula for the induced stress, we need consider only the first order interaction. The "equation of motion" of the state-vector of the meson field therefore takes the form

$$i\hbar c \frac{\delta \Psi[\sigma]}{\delta\sigma(x)} = -\frac{1}{2} h_{\mu\nu}(x)\, \mathring{\Theta}_{M\mu\nu}(x)\, \Psi[\sigma], \tag{3.1}$$

of which the first order solution, with appropriate boundary condition, is

$$\Psi[\sigma] = \left[1 + \frac{i}{2\hbar c}\int_{-\infty}^{\sigma} h_{\mu\nu}(x')\, \mathring{\Theta}_{M\mu\nu}(x')\, d\omega'\right] \Psi_0 \tag{3.2}$$

where Ψ_0 denotes the vacuum state-vector.

To say that the gravitational field is externally impressed is to say that the geometry of <u>real space</u> is externally impressed. In order to obtain a physically significant result we must view the induced mesic stress with respect to this impressed <u>real space</u> rather than with respect to the mathematical background space. For this purpose we must use the full stress tensor $\Theta_{M\mu\nu}$ in its covariant form. Using equations (I, 3.40) and (2.23) we find that this tensor takes the following form in the interaction representation.

$$\begin{aligned}\Theta_{M\mu\nu}[x,\sigma] &= U[\sigma]\,\Theta_{M\mu\nu}(x)\,U^*[\sigma] \\ &= U[\sigma]\Big[\tfrac{1}{2}\{\phi_{,\mu},\phi_{,\nu}\} - \tfrac{1}{2}g_{\mu\nu}\big(g^{\sigma\tau}\phi_{,\sigma}\phi_{,\tau} + \kappa^2\phi^2 - \tfrac{1}{4}\hbar c\kappa^2\Delta^{(1)}(0)\big)\Big](x)\,U^*[\sigma] \\ &= \overset{\circ}{\Theta}_{M\mu\nu}(x) + \Big[-\tfrac{1}{2}n_\mu n_\alpha \phi_{\alpha\beta}\{\phi_{,\nu},\phi_{,\beta}\} - \tfrac{1}{2}n_\nu n_\alpha\phi_{\alpha\beta}\{\phi_{,\mu},\phi_{\beta}\} + \tfrac{1}{2}\delta_{\mu\nu}n_\sigma n_\alpha\phi_{\alpha\beta}\{\phi_{,\sigma},\phi_{,\beta}\} \\ &\quad - \tfrac{1}{2}(\phi_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\phi_{\lambda\lambda})(\phi_{,\sigma}\phi_{,\sigma} + \kappa^2\phi^2 - \tfrac{1}{4}\hbar c\kappa^2\Delta^{(1)}(0)) \\ &\quad + \tfrac{1}{4}\delta_{\mu\nu}(\phi_{\sigma\tau} - \tfrac{1}{2}\delta_{\sigma\tau}\phi_{\rho\rho})\{\phi_{,\sigma},\phi_{,\tau}\}\Big][x,\sigma] + \cdots\cdots \end{aligned} \tag{3.2}$$

Making use of the equation $\langle\{\phi(x),\phi(x')\}\rangle_0 = \hbar c\,\Delta^{(1)}(x-x')$, and the fact that $\Delta^{(1)}_{,\mu\nu}(0) = \frac{1}{4}\kappa^2\delta_{\mu\nu}\Delta^{(1)}(0)$, we obtain for the vacuum expectation value of this tensor,

$$\langle\Theta_{M\mu\nu}[x,\sigma]\rangle_0 = \tfrac{1}{8}\hbar c\kappa^2\Delta^{(1)}(0)\big(n_\mu n_\alpha\phi_{\alpha\nu} + n_\nu n_\alpha\phi_{\alpha\mu} - \delta_{\mu\nu}n_\alpha n_\beta\phi_{\alpha\beta} + \phi_{\mu\nu}\big)[x,\sigma] + \cdots \tag{3.3}$$

It may perhaps immediately be objected that this is a divergent expression. It should be remembered, however, that the vacuum state Ψ_0, with respect to which this expectation value is taken, is a "false state" for the problem at hand. We are here interested in the expectation value of $\Theta_{M\mu\nu}$ in the actual state $\Psi[\sigma]$, which differs from the vacuum state through the presence of the gravitational field. In the calculation of the true expectation value, it will be found that the divergent quantity on the right in (3.3) is exactly cancelled by other terms.

To first order, we have

$$\langle\Theta_{M\mu\nu}[x,\sigma]\rangle = \langle\Theta_{M\mu\nu}[x,\sigma]\rangle_0 + \frac{i}{2\hbar c}\int_{-\infty}^{\sigma}\langle[\overset{\circ}{\Theta}_{M\mu\nu}(x),\overset{\circ}{\Theta}_{M\sigma\tau}(x')]\rangle_0\,h_{\sigma\tau}(x')\,d\omega', \tag{3.4}$$

and we are led to evaluate the vacuum expectation value of the indicated commutator. It is convenient at this point to record the explicit expressions, not only for the commutator in (3.4), but of other commutator and anticommutator expressions which will be used in this and the following section. By well known methods one may obtain the following expressions.

$$\langle[\mathring{\Theta}_{M\mu\nu}(x),\ \mathring{\Theta}_{M\sigma\tau}(x')]\rangle_0$$
$$= \frac{i\hbar^2c^2}{2}\Big[\Delta_{,\mu\sigma}\Delta^{(1)}_{,\nu\tau} + \Delta_{,\nu\tau}\Delta^{(1)}_{,\mu\sigma} + \Delta_{,\mu\tau}\Delta^{(1)}_{,\nu\sigma} + \Delta_{,\nu\sigma}\Delta^{(1)}_{,\mu\tau}$$
$$-\delta_{\sigma\tau}\left(\Delta_{,\alpha\mu}\Delta^{(1)}_{,\alpha\nu} + \Delta_{,\alpha\nu}\Delta^{(1)}_{,\alpha\mu} + K^2\Delta_{,\mu}\Delta^{(1)}_{,\nu} + K^2\Delta_{,\nu}\Delta^{(1)}_{,\mu}\right)$$
$$-\delta_{\mu\nu}\left(\Delta_{,\alpha\sigma}\Delta^{(1)}_{,\alpha\tau} + \Delta_{,\alpha\tau}\Delta^{(1)}_{,\alpha\sigma} + K^2\Delta_{,\sigma}\Delta^{(1)}_{,\tau} + K^2\Delta_{,\tau}\Delta^{(1)}_{,\sigma}\right)$$
$$+\delta_{\mu\nu}\delta_{\sigma\tau}\left(\Delta_{,\alpha\beta}\Delta^{(1)}_{,\alpha\beta} + 2K^2\Delta_{,\alpha}\Delta^{(1)}_{,\alpha} + K^4\Delta\Delta^{(1)}\right)\Big](x-x') \tag{3.5}$$

$$[\mathring{\Theta}_{M\mu\nu}(x),\ \mathring{\Theta}_{M\mu\nu}(x')]_1 = -2i\hbar c\Big[\Delta_{,\alpha\beta}(x-x')\left(\phi_{,\alpha}(x)\,\phi_{,\beta}(x')\right)_1 - K^2\Delta_{,\alpha}(x-x')\left(\phi_{,\alpha}(x)\,\phi(x')\right)_1$$
$$+K^2\Delta_{,\alpha}(x-x')\left(\phi(x)\,\phi_{,\alpha}(x')\right)_1 + K^2\Delta(x-x')\left(\phi_{,\alpha}(x)\,\phi_{,\alpha}(x')\right)_1$$
$$-2K^4\Delta(x-x')\left(\phi(x)\,\phi(x')\right)_1\Big] \tag{3.6}$$

$$[\mathring{\Theta}_{M\mu\mu}(x),\ \mathring{\Theta}_{M\sigma\sigma}(x')]_1 = -4i\hbar c\Big[\Delta_{,\alpha\beta}(x-x')\left(\phi_{,\alpha}(x)\,\phi_{,\beta}(x')\right)_1 - 2K^2\Delta_{,\alpha}(x-x')\left(\phi_{,\alpha}(x)\,\phi(x')\right)_1$$
$$+2K^2\Delta_{,\alpha}(x-x')\left(\phi(x)\,\phi_{,\alpha}(x')\right)_1 - 4K^4\Delta(x-x')\left(\phi(x)\,\phi(x')\right)_1\Big] \tag{3.7}$$

$$\langle\{\mathring{\Theta}_{M\mu\nu}(x),\ \mathring{\Theta}_{M\mu\nu}(x')\}\rangle_0 = -\frac{\hbar^2c^2}{2}\Big[\Delta_{,\alpha\beta}\Delta_{,\alpha\beta} - \Delta^{(1)}_{,\alpha\beta}\Delta^{(1)}_{,\alpha\beta} + 2K^2\left(\Delta_{,\alpha}\Delta_{,\alpha} - \Delta^{(1)}_{,\alpha}\Delta^{(1)}_{,\alpha}\right)$$
$$+3K^4\left(\Delta\Delta - \Delta^{(1)}\Delta^{(1)}\right)\Big](x-x') \tag{3.8}$$

$$\langle\{\mathring{\Theta}_{M\mu\mu}(x),\ \mathring{\Theta}_{M\sigma\sigma}(x')\}\rangle_0 = -\hbar^2c^2\Big[\Delta_{,\alpha\beta}\Delta_{,\alpha\beta} - \Delta^{(1)}_{,\alpha\beta}\Delta^{(1)}_{,\alpha\beta} + 4K^2\left(\Delta_{,\alpha}\Delta_{,\alpha} - \Delta^{(1)}_{,\alpha}\Delta^{(1)}_{,\alpha}\right)$$
$$+4K^4\left(\Delta\Delta - \Delta^{(1)}\Delta^{(1)}\right)\Big](x-x') \tag{3.9}$$

$$\{\mathring{\Theta}_{M\mu\nu}(x),\ \mathring{\Theta}_{M\mu\nu}(x')\}_1 = -2\hbar c\Big[\Delta^{(1)}_{,\alpha\beta}(x-x')\left(\phi_{,\alpha}(x)\,\phi_{,\beta}(x')\right)_1 - K^2\Delta^{(1)}_{,\alpha}(x-x')\left(\phi_{,\alpha}(x)\,\phi(x')\right)_1$$
$$+K^2\Delta^{(1)}_{,\alpha}(x-x')\left(\phi(x)\,\phi_{,\alpha}(x')\right)_1 + K^2\Delta^{(1)}(x-x')\left(\phi_{,\alpha}(x)\,\phi_{,\alpha}(x')\right)_1$$
$$-2K^4\Delta^{(1)}(x-x')\left(\phi(x)\,\phi(x')\right)_1\Big] \tag{3.10}$$

$$\{\mathring{\Theta}_{M\mu\mu}(x),\ \mathring{\Theta}_{M\sigma\sigma}(x')\}_1 = -4\hbar c\Big[\Delta^{(1)}_{,\alpha\beta}(x-x')\left(\phi_{,\alpha}(x)\,\phi_{,\beta}(x')\right)_1 - 2K^2\Delta^{(1)}_{,\alpha}(x-x')\left(\phi_{,\alpha}(x)\,\phi(x')\right)_1$$
$$+2K^2\Delta^{(1)}_{,\alpha}(x-x')\left(\phi(x)\,\phi_{,\alpha}(x')\right)_1 - 4K^4\Delta^{(1)}(x-x')\left(\phi(x)\,\phi(x')\right)_1\Big] \tag{3.11}$$

The subscript 1 denotes the one-particle part of the indicated operator. It should be noted that $(\phi(x) \quad \phi(x'))_1 = (\phi(x') \quad \phi(x))_1$, etc.

An important requirement which must be met by expression (3.4) is that of G-gauge invariance. In the absence of a real gravitational field the induced stress must vanish. That is, if $h_{\mu\nu} = \delta\Lambda_{\mu,\nu} + \delta\Lambda_{\nu,\mu}$ then $\langle \Theta_{M\mu\nu}[x,\sigma] \rangle$ must be zero. In the present example we cannot use an argument analagous to that used by Schwinger[11] in his treatment of vacuum polarisation,

[11] Op. cit.

concerning the commutability of the current with a time-like component of itself at the same point, since integrals of the form $\int_\sigma [\mathring{\Theta}_{M\mu\nu}(x), \mathring{\Theta}_{M\sigma\tau}(x')] f(x') d\sigma'_\tau$ do not, in general, vanish. We must, instead, carry out a direct calculation. The reduction of expression (3.4) for the case $h_{\mu\nu} = \delta\Lambda_{\mu,\nu} + \delta\Lambda_{\nu,\mu}$ is straightforward but tedious, and will be omitted here. One makes use of equations (A.7) and (A.8) of the Appendix and the fact that $\Delta^{(1)}_{,\mu\nu}(0) = \frac{1}{4}\kappa^2 \delta_{\mu\nu} \Delta^{(1)}(0)$. The terms in (3.4) contributed by $\langle \Theta_{M\mu\nu}[x,\sigma] \rangle_0$ are cancelled by terms which the integral generates. The integral itself is greatly reducible, and one finds

$$\begin{aligned}
\langle \Theta_{M\mu\nu}[x,\sigma] \rangle &= \langle \Theta_{M\mu\nu}[x,\sigma] \rangle_0 + \frac{i}{\hbar c} \int_{-\infty}^{\sigma} \frac{\partial}{\partial x'_\tau} \Big[\langle [\mathring{\Theta}_{M\mu\nu}(x), \mathring{\Theta}_{M\sigma\tau}(x')] \rangle_0 \, \delta\Lambda_\sigma(x') \Big] d\omega' \\
&= \tfrac{1}{8} \hbar c \kappa^2 \Delta^{(1)}(0) \Big[n_\mu n_\alpha (\delta\Lambda_{\alpha,\nu} + \delta\Lambda_{\nu,\alpha} - \delta_{\alpha\nu}\delta\Lambda_{\sigma,\sigma}) + n_\nu n_\alpha (\delta\Lambda_{\alpha,\mu} + \delta\Lambda_{\mu,\alpha} - \delta_{\alpha\mu}\delta\Lambda_{\sigma,\sigma}) \\
&\qquad - \delta_{\mu\nu} n_\alpha n_\beta (\delta\Lambda_{\alpha,\beta} + \delta\Lambda_{\beta,\alpha} - \delta_{\alpha\beta}\delta\Lambda_{\sigma,\sigma}) + \delta\Lambda_{\mu,\nu} + \delta\Lambda_{\nu,\mu} - \delta_{\mu\nu}\delta\Lambda_{\sigma,\sigma} \Big][x,\sigma] \\
&\quad + \frac{i}{\hbar c} \int_\sigma \langle [\mathring{\Theta}_{M\mu\nu}(x), \mathring{\Theta}_{M\sigma\tau}(x')] \rangle_0 \, \delta\Lambda_\sigma(x') \, d\sigma'_\tau \\
&= -\hbar c \, \Delta^{(1)}_{,\mu\nu\sigma}(0) \, \delta\Lambda_\sigma(x). \qquad (3.12)
\end{aligned}$$

Here we come upon a generalization, to third derivatives, of the famous photon self-energy ambiguity of quantum electrodynamics. The requirements of G-gauge invariance evidently lead us to make the formal identification[12] footnote on next page

$$\Delta^{(1)}_{,\mu\nu\sigma}(0) = 0. \qquad (3.13)$$

12 In electrodynamics, the analogous identification is $\Delta^{(1)}_{,\mu}(0) = 0$.

The reasonableness of this identification is clear from the symmetry properties of the function $\Delta^{(1)}$, and it can also be justified through use of the "invariant regularization" procedure of Pauli and Villars[13].

13 Rev. Mod. Phys. 21, 434 (1949).

Adopting the restriction that no real meson pairs be produced in the vacuum by the gravitational field, which is equivalent to the statement

$$\int_{-\infty}^{\infty} h_{\mu\nu}(x')\,\overset{\circ}{\Theta}_{M\mu\nu}(x')\,d\omega'\;\Psi_0 = 0, \tag{3.14}$$

we may rewrite equation (3.4) in the form

$$\langle \Theta_{M\mu\nu}[x,\sigma]\rangle = \langle \Theta_{M\mu\nu}[x,\sigma]\rangle_0 + \frac{i}{4\hbar c}\int_{-\infty}^{\infty}\langle[\overset{\circ}{\Theta}_{M\mu\nu}(x), \overset{\circ}{\Theta}_{M\sigma\tau}(x')]\rangle_0\,\varepsilon[\sigma,x']\,h_{\sigma\tau}(x')\,d\omega' \tag{3.15}$$

where $\varepsilon[\sigma, x']$ equals + 1 or − 1 according as x' lies to the past or to the future of the surface σ. Making use of equation (A.10) of the Appendix, we find that the terms involving surface normals in the expression for $\langle \Theta_{M\mu\nu}[x,\sigma]\rangle_0$ are exactly cancelled by corresponding terms in the integral, and we are left with

$$\begin{aligned}\langle \Theta_{M\mu\nu}[x,\sigma]\rangle = {} & \tfrac{1}{8}\hbar c\,\kappa^2\Delta^{(1)}(0)\,\phi_{\mu\nu}(x) \\ & + \frac{\hbar c}{4}\int\Big[\bar\Delta_{,\mu\sigma}\Delta^{(1)}_{,\nu\tau} + \bar\Delta_{,\nu\tau}\Delta^{(1)}_{,\mu\sigma} + \bar\Delta_{,\mu\tau}\Delta^{(1)}_{,\nu\sigma} + \bar\Delta_{,\nu\sigma}\Delta^{(1)}_{,\mu\tau} \\ & - \delta_{\sigma\tau}\big(\bar\Delta_{,\alpha\mu}\Delta^{(1)}_{,\alpha\nu} + \bar\Delta_{,\alpha\nu}\Delta^{(1)}_{,\alpha\mu} + \kappa^2\bar\Delta_{,\mu}\Delta^{(1)}_{,\nu} + \kappa^2\bar\Delta_{,\nu}\Delta^{(1)}_{,\mu}\big) \\ & - \delta_{\mu\nu}\big(\bar\Delta_{,\alpha\sigma}\Delta^{(1)}_{,\alpha\tau} + \bar\Delta_{,\alpha\tau}\Delta^{(1)}_{,\alpha\sigma} + \kappa^2\bar\Delta_{,\sigma}\Delta^{(1)}_{,\tau} + \kappa^2\bar\Delta_{,\tau}\Delta^{(1)}_{,\sigma}\big) \\ & + \delta_{\mu\nu}\delta_{\sigma\tau}\big(\bar\Delta_{,\alpha\beta}\Delta^{(1)}_{,\alpha\beta} + 2\kappa^2\bar\Delta_{,\alpha}\Delta^{(1)}_{,\alpha} + \kappa^4\bar\Delta\Delta^{(1)}\big)\Big](x-x')\,h_{\sigma\tau}(x')\,d\omega'\end{aligned} \tag{3.16}$$

By use of the equation $(\Box^2 - \kappa^2)\bar\Delta(x) = -\delta(x)$ the first term in (3.16) may be absorbed into the integral, so that we may write

$$\langle \Theta_{M\mu\nu}[x,\sigma]\rangle = \frac{\hbar c}{4}\int G_{\mu\nu\sigma\tau}(x-x')\,h_{\sigma\tau}(x')\,d\omega', \tag{3.17}$$

where

$$
\begin{aligned}
G_{\mu\nu\sigma\tau} = {} & \bar{\Delta}_{,\mu\sigma}\Delta^{(1)}_{,\nu\tau} + \bar{\Delta}_{,\nu\tau}\Delta^{(1)}_{,\mu\sigma} + \bar{\Delta}_{,\mu\tau}\Delta^{(1)}_{,\nu\sigma} + \bar{\Delta}_{,\nu\sigma}\Delta^{(1)}_{,\mu\tau} \\
& - \delta_{\sigma\tau}\left(\bar{\Delta}_{,\alpha\mu}\Delta^{(1)}_{,\alpha\nu} + \bar{\Delta}_{,\alpha\nu}\Delta^{(1)}_{,\alpha\mu} + K^2\bar{\Delta}_{,\mu}\Delta^{(1)}_{,\nu} + K^2\bar{\Delta}_{,\nu}\Delta^{(1)}_{,\mu}\right) \\
& - \delta_{\mu\nu}\left(\bar{\Delta}_{,\alpha\sigma}\Delta^{(1)}_{,\alpha\tau} + \bar{\Delta}_{,\alpha\tau}\Delta^{(1)}_{,\alpha\sigma} + K^2\bar{\Delta}_{,\sigma}\Delta^{(1)}_{,\tau} + K^2\bar{\Delta}_{,\tau}\Delta^{(1)}_{,\sigma}\right) \\
& + \delta_{\mu\nu}\delta_{\sigma\tau}\left(\bar{\Delta}_{,\alpha\beta}\Delta^{(1)}_{,\alpha\beta} + 2K^2\bar{\Delta}_{,\alpha}\Delta^{(1)}_{,\alpha} + \tfrac{1}{4}K^2\bar{\Delta}_{,\alpha\alpha}\Delta^{(1)} + \tfrac{1}{4}K^2\bar{\Delta}\Delta^{(1)}_{,\alpha\alpha} + \tfrac{1}{2}K^4\bar{\Delta}\Delta^{(1)}\right) \\
& - \left(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma}\right)\left(\tfrac{1}{4}K^2\bar{\Delta}_{,\alpha\alpha}\Delta^{(1)} + \tfrac{1}{4}K^2\bar{\Delta}\Delta^{(1)}_{,\alpha\alpha} - \tfrac{1}{2}K^4\bar{\Delta}\Delta^{(1)}\right)
\end{aligned}
\tag{3.18}
$$

in which vanishing terms proportional to $(\Box^2 - K^2)\Delta^{(1)}$ have been added so as to make the expression symmetrical.

If the Fourier integral representations[14] of the functions $\bar{\Delta}$ and $\Delta^{(1)}$

[14] J. Schwinger, op. cit.

are used, $G_{\mu\nu\sigma\tau}$ may be expressed in the following form

$$
\begin{aligned}
G_{\mu\nu\sigma\tau}(x) = \frac{1}{2(2\pi)^7}\iint\Big[& k'_\mu k'_\sigma k''_\nu k''_\tau + k'_\nu k'_\tau k''_\mu k''_\sigma + k'_\mu k'_\tau k''_\nu k''_\sigma + k'_\nu k'_\sigma k''_\mu k''_\tau \\
& - \delta_{\sigma\tau}\left(k'_\alpha k''_\alpha k'_\mu k''_\nu + k'_\alpha k''_\alpha k'_\nu k''_\mu - K^2 k'_\mu k''_\nu - K^2 k'_\nu k''_\mu\right) \\
& - \delta_{\mu\nu}\left(k'_\alpha k''_\alpha k'_\sigma k''_\tau + k'_\alpha k''_\alpha k'_\tau k''_\sigma - K^2 k'_\sigma k''_\tau - K^2 k'_\tau k''_\sigma\right) \\
& + \delta_{\mu\nu}\delta_{\sigma\tau}\left(k'_\alpha k'_\beta k''_\alpha k''_\beta - 2K^2 k'_\alpha k''_\alpha - \tfrac{1}{4}K^2 k'^2 - \tfrac{1}{4}K^2 k''^2 + \tfrac{1}{2}K^4\right) \\
& + \left(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma}\right)\left(\tfrac{1}{4}K^2 k'^2 + \tfrac{1}{4}K^2 k''^2 + \tfrac{1}{2}K^4\right)\Big]\left[\frac{\delta(k''^2+K^2)}{k'^2+K^2} + \frac{\delta(k'^2+K^2)}{k''^2+K^2}\right] e^{i(k'+k'')x}\,(dk')(dk'').
\end{aligned}
\tag{3.19}
$$

When $\langle \Theta_{M\,\mu\nu}[x,\sigma] \rangle$ is expressed in the form (3.17) it is easy to show that the law of conservation of energy and momentum in the meson field is satisfied by the induced stress. To the first order, we may write

$$
g^{\nu\sigma}\langle \Theta_{M\,\mu\nu}[x,\sigma] \rangle_{;\sigma} = \langle \Theta_{M\,\mu\nu}[x,\sigma] \rangle_{,\nu} = \frac{\hbar c}{4}\int G_{\mu\nu\sigma\tau,\nu}(x-x')\,h_{\sigma\tau}(x')\,d\omega'. \tag{3.20}
$$

The requirements of energy-momentum conservation and of G-gauge invariance are seen to be identical and to depend on the vanishing of the divergence of

$G_{\mu\nu\sigma\tau}$. That $G_{\mu\nu\sigma\tau,\tau}$ does, indeed, vanish may be demonstrated either directly from expression (3.18), by making use of (3.13) and the equations $(\Box^2-\kappa^2)\bar{\Delta}(x)=-\delta(x)$, $(\Box^2-\kappa^2)\Delta^{(1)}=0$ and $\Delta^{(1)}_{,\mu\nu}(0)=\frac{1}{4}\kappa^2\delta_{\mu\nu}\Delta^{(1)}(0)$, or from expression (3.19), by introducing the variable of integration $k=k'+k''$ and using the equations $(k''^2+\kappa^2)\delta(k''^2+\kappa^2)=0$, $\int k''_\mu k''_\nu\,\delta(k''^2+\kappa^2)(dk'')=\frac{1}{4}\delta_{\mu\nu}\int k''^2\delta(k''^2+\kappa^2)(dk'')$, and $\int k''_\mu k''_\nu k''_\sigma\,\delta(k''^2+\kappa^2)(dk'')=0$. In either case the reduction is straightforward but tedious.

Another important identity satisfied by $G_{\mu\nu\sigma\tau}$, which may be verified by direct calculation from (3.18), is the following.

$$G_{\mu\mu\nu\nu}(x)-2G_{\mu\nu\mu\nu}(x)=\kappa^2(\Box^2-\kappa^2)\left[4\bar{\Delta}(x)\Delta^{(1)}(x)+2\bar{\Delta}(x)\Delta^{(1)}(0)\right]. \qquad (3.21)$$

In order to reduce the kernel $G_{\mu\nu\sigma\tau}$ to a more usable explicit form it is convenient to make, in (3.19), the following transformation in the variables of integration.

$$\left.\begin{array}{ll} & k=k'+k'', \quad p=\frac{v}{2}(k'+k'')+\frac{1}{2}(k'-k'') \\ \text{with inverse} & k'=\frac{1-v}{2}k+p, \quad k''=\frac{1+v}{2}k-p \\ \text{and Jacobian} & \begin{vmatrix}\frac{1-v}{2} & 1\\ \frac{1+v}{2} & -1\end{vmatrix}^4=1\end{array}\right\} \qquad (3.22)$$

where v is a parameter introduced through the identity[15]

$$\frac{\delta(k''^2+\kappa^2)}{k'^2+\kappa^2}+\frac{\delta(k'^2+\kappa^2)}{k''^2+\kappa^2}=\frac{\delta(k'^2+\kappa^2)-\delta(k''^2+\kappa^2)}{k''^2-k'^2}$$

$$=-\frac{1}{2}\int_{-1}^{1}\delta'\left(\frac{k'^2+k''^2}{2}+\frac{k'^2-k''^2}{2}v+\kappa^2\right)dv=-\frac{1}{2}\int_{-1}^{1}\delta'\left(p^2+\frac{1-v^2}{4}k^2+\kappa^2\right)dv. \qquad (3.23)$$

[15] J. Schwinger, Phys. Rev. 76, 790 (1949).

Inserting these new variables into (3.19), making use of (3.23), and, in view of the symmetry in the resulting integration over p, replacing factors of the forms $p_\mu p_\nu$ and $p_\mu p_\nu p_\sigma p_\tau$ by the forms $\frac{1}{4}\delta_{\mu\nu}p^2$ and $\frac{1}{24}(\delta_{\mu\nu}\delta_{\sigma\tau}+\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma})p^4$ respectively and dropping terms in the integrand which are linear and cubic in p, we obtain, after collecting terms, the following form for $G_{\mu\nu\sigma\tau}$.

$$G_{\mu\nu\sigma\tau} = (\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma})(A+K^2A')+\delta_{\mu\nu}\delta_{\sigma\tau}(B+K^2B')-\delta_{\mu\sigma}(C+K^2C')_{,\nu\tau}-\delta_{\nu\tau}(C+K^2C')_{,\mu\sigma}$$
$$-\delta_{\mu\tau}(C+K^2C')_{,\nu\sigma}-\delta_{\nu\sigma}(C+K^2C')_{,\mu\tau}-\delta_{\mu\nu}(D+K^2D')_{,\sigma\tau}-\delta_{\sigma\tau}(D+K^2D')_{,\mu\nu}+E_{,\mu\nu\sigma\tau} \qquad (3.24)$$

where, of the five functions $A+K^2A'$, $B+K^2B'$, $C+K^2C'$, $D+K^2D'$ and E, the only ones which are of interest to us here are $D+K^2D'$ and E. These two functions have the respective forms

$$D(x)+K^2D'(x) = \frac{1}{4(2\pi)^7}\int_{-1}^{1}\iint\left[\frac{(1-v^2)^2}{8}k^2+\frac{3v^2-1}{4}p^2-\frac{1-v^2}{2}K^2\right]\delta'\left(p^2+\frac{1-v^2}{4}k^2+K^2\right)e^{ikx}(dk)(dp)\,dv \qquad (3.25)$$

$$E(x) = -\frac{1}{4(2\pi)^7}\int_{-1}^{1}\iint\frac{(1-v^2)^2}{4}\delta'\left(p^2+\frac{1-v^2}{4}k^2+K^2\right)e^{ikx}(dk)(dp)\,dv. \qquad (3.26)$$

The reason for implicitly separating out a part which vanishes as $K \to 0$, in each of the first four functions occurring in (3.24), will become clear in a moment.

The change in variables of integration (3.22) has destroyed that symmetry in the momentum - space integrals which originally enabled us to infer the identity $G_{\mu\nu\sigma\tau,\tau} = 0$. The identity (3.21) also can no longer be directly inferred from the integral expression for $G_{\mu\nu\sigma\tau}$. If the functions $A+K^2A'$, $B+K^2B'$, etc. were expressed in terms of convergent integrals, this would make no difference. But since these integrals are <u>actually</u> <u>divergent</u>, and hence ambiguous, we must, in order to preserve G-gauge invariance and the algebraic symmetry properties of $G_{\mu\nu\sigma\tau}$, re-impose these identities as conditions upon the functions $A+K^2A'$, $B+K^2B'$, etc.[16]. We set

---

[16] This procedure is really equivalent to a more elegant one which consists of "preparing" the integrand of (3.19) in advance of the introduction of the new variables of integration, by judiciously adding terms which vanish like $\int k_\mu k_\nu k_\sigma \delta(k^2+K^2)(dk)$ in such a way that the divergence condition and other symmetry conditions become identically satisfied independently of such momentum-space integrals. The present method avoids the necessity of having to juggle terms, and gives the same result with less effort.

$$0 = G_{\mu\nu\sigma\tau,\tau} = \delta_{\mu\sigma}(A+\kappa^2A')_{,\nu} + \delta_{\nu\sigma}(A+\kappa^2A')_{,\mu} + \delta_{\mu\nu}(B+\kappa^2B')_{,\sigma} - \delta_{\mu\sigma}(C+\kappa^2C')_{,\nu\tau\tau} - 2(C+\kappa^2C')_{,\mu\nu\sigma}$$
$$- \delta_{\nu\sigma}(C+\kappa^2C')_{,\mu\tau\tau} - \delta_{\mu\nu}(D+\kappa^2D')_{,\sigma\tau\tau} - (D+\kappa^2D')_{,\mu\nu\sigma} + E_{,\mu\nu\sigma\tau\tau} \tag{3.27}$$

$$\kappa^2(\Box^2-\kappa^2)(4\bar{\Delta}\Delta^{(1)} + 2\bar{\Delta}\Delta^{(1)}(0)) = G_{\mu\mu\nu\nu} - 2G_{\mu\nu\mu\nu}$$
$$= -32(A+\kappa^2A') + 8(B+\kappa^2B') + 16(C+\kappa^2C')_{,\alpha\alpha} - 4(D+\kappa^2D')_{,\alpha\alpha} - E_{,\alpha\alpha\beta\beta}\,. \tag{3.28}$$

The separation of the functions $A+\kappa^2A'$, $B+\kappa^2B'$, etc. into parts which vanish as $\kappa \to 0$ and parts which do not is made unique by requiring that the relations which hold between the various parts A, B, C, D, E, A', B', C', D' be independent of κ. Comparing corresponding terms in (3.27) and (3.28), we obtain the equations

$$A - \Box^2 C = 0, \quad B - \Box^2 D = 0, \quad -2C - D + \Box^2 E = 0, \quad -32A + 8B + 16\Box^2 C - 4\Box^2 D - \Box^4 E = 0 \tag{3.29}$$

$$A' - \Box^2 C' = 0, \quad B' - \Box^2 D' = 0, \quad -2C' - D' = 0 \tag{3.30}$$

of which the solutions are

$$8A = \tfrac{4}{3}B = 8\Box^2 C = \tfrac{4}{3}\Box^2 D = \Box^4 E\,, \tag{3.31}$$

$$-2A' = B' = -2\Box^2 C' = \Box^2 D'\,. \tag{3.32}$$

We may now write

$$G_{\mu\nu\sigma\tau} = \tfrac{1}{8}\Big[(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma})E_{,\alpha\alpha\beta\beta} - \delta_{\mu\sigma}E_{,\alpha\alpha\nu\tau} - \delta_{\nu\tau}E_{,\alpha\alpha\mu\sigma} - \delta_{\mu\tau}E_{,\alpha\alpha\nu\sigma} - \delta_{\nu\sigma}E_{,\alpha\alpha\mu\tau}$$
$$+ 2\delta_{\sigma\tau}E_{,\alpha\alpha\mu\nu} + 2\delta_{\mu\nu}E_{,\alpha\alpha\sigma\tau} - 2\delta_{\mu\nu}\delta_{\sigma\tau}E_{,\alpha\alpha\beta\beta}\Big] + \Big[E_{,\mu\nu\sigma\tau} - \delta_{\sigma\tau}E_{,\alpha\alpha\mu\nu}\Big]$$
$$- \delta_{\mu\nu}\Big[E_{,\alpha\alpha\sigma\tau} - \delta_{\sigma\tau}E_{,\alpha\alpha\beta\beta}\Big] - \frac{\kappa^2}{2}\Big[(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma})D'_{,\alpha\alpha\beta\beta}$$
$$- \delta_{\mu\sigma}D'_{,\alpha\alpha\nu\tau} - \delta_{\nu\tau}D'_{,\alpha\alpha\mu\sigma} - \delta_{\mu\tau}D'_{,\alpha\alpha\nu\sigma} - \delta_{\nu\sigma}D'_{,\alpha\alpha\mu\tau}$$
$$+ 2\delta_{\sigma\tau}D'_{,\alpha\alpha\mu\nu} + 2\delta_{\mu\nu}D'_{,\alpha\alpha\sigma\tau} - 2\delta_{\mu\nu}\delta_{\sigma\tau}D'_{,\alpha\alpha\beta\beta}\Big] \tag{3.33}$$

Inserting this expression into (3.17), carrying out repeated partial integrations, and referring to equation (I, 5.11) which expresses the tensor $\overset{1}{G}_{\mu\nu}$ in terms of the derivatives of $h_{\mu\nu}$, we find

$$\langle \Theta_{M\mu\nu}[x,\sigma]\rangle = \frac{\hbar c}{4}\int\Big[E(x-x')\big(\tfrac{1}{2}\overset{1}{G}_{\mu\nu,\alpha\alpha} + \overset{1}{G}_{\alpha\alpha,\mu\nu} - \delta_{\mu\nu}\overset{1}{G}_{\alpha\alpha,\beta\beta}\big)(x')$$
$$- 2\kappa^2 D'(x-x')\overset{1}{G}_{\mu\nu}(x')\Big]\,d\omega' \tag{3.34}$$

Expression (3.34) is manifestly G-gauge invariant to first order, since the tensor $\overset{1}{G}_{\mu\nu}$ is invariant under the linear G-gauge transformation (1.12)[17].

[17] It may be noted that a gravitational field quantity must be of at least the second differential order in the potentials $\phi_{\mu\nu}$ in order that it be invariant under the transformation (1.12). This is a first order expression of the well-known fact that any non trivial fundamental tensor in Riemannian geometry must be of at least the second differential order in the metric tensor.

Equation (3.34) may be cast in a different form, by making use of the Einstein field equations. It may be shown[18] that the tensor $G^{\mu\nu}$ satisfies

[18] B. S. DeWitt unpublished.

the identity

$$\tfrac{1}{2} g^{\frac{1}{2}} \left(G_\mu{}^\nu + G_\nu{}^\mu \right) \equiv \overset{1}{G}_{\mu\nu} + \tfrac{\beta}{2} \Theta_{G\,\mu\nu} \tag{3.35}$$

where $\Theta_{G\,\mu\nu}$ is the symmetric stress tensor of the gravitational field, defined by (I, 4.10). This is a useful identity, for it connects quantities which are covariant with quantities which are not. Since the Einstein field equations enable us to replace $G^{\mu\nu}$ by $-\frac{\beta}{2}\Theta_M{}^{\mu\nu}$, equation (3.35) may be written in the form

$$\overset{1}{G}_{\mu\nu} = -\tfrac{\beta}{2}\left(\Theta_{G\,\mu\nu} + \Theta'_{M\,\mu\nu} \right) = -\tfrac{\beta}{2}\Theta_{\mu\nu} \tag{3.36}$$

where $\Theta'_{M\,\mu\nu}$ is defined by (I, 4.12) and $\Theta_{\mu\nu}$ is the symmetric stress tensor of the combined matter and gravitational fields. Equation (3.36) clearly shows the "feedback" characteristics of the gravitational field: the gravitational field is partially generated by its own stress. We may now write

$$\langle \Theta_{M\,\mu\nu}[x,\sigma] \rangle = -\frac{\hbar\beta c}{8} \iint \left[E(x-x')\left(\tfrac{1}{2}\Theta^{EXT}_{\mu\nu,\alpha\alpha} + \Theta^{EXT}_{\alpha\alpha,\mu\nu} - \delta_{\mu\nu}\Theta^{EXT}_{\alpha\alpha,\rho\rho} \right)(x') - 2K^2 D'(x-x')\Theta^{EXT}_{\mu\nu}(x') \right] d\omega' \tag{3.37}$$

where $\Theta^{EXT}_{\mu\nu}$ denotes the total external stress, gravitational as well as matter, which generates the gravitational field.

The feedback properties of the gravitational field make it impossible for us to say, in analogy with the case of the photon in electrodynamics, that a gravitational quantum has no self-energy arising from its coupling with a matter field. A gravitational quantum carries stress, and this stress will itself induce a non-vanishing matter stress in the vacuum. This type of self-interaction is over and above the ordinary self-interaction described by $\mathcal{H}_G^{SELF}[x,\sigma]$ which is always present. In spite of these self-interactions, and divergent though they may be, the existence of the G-gauge group makes it reasonable to suppose that they will not correspond to an actual gravitational "self-mass" but will always be interpretable on the basis of a simple "renormalization" of the gravitational potentials.

The induced stress of (3.37) cannot be interpreted as expressing a simple renormalization of the external stress, for both terms in the integrand are logarithmically divergent and the first is structure dependent.

The explicit form of the kernel $D'(x-x')$ may be obtained by making use of the relation $x\,\delta'(x) = -\delta(x)$ and carrying out a partial integration on (3.25). One finds

$$D(x)+\kappa^2 D'(x) = \frac{1}{4(2\pi)^7}\int_{-1}^{1}\iint\left[\frac{3}{16}(1-v^2)^2 k^2 - \frac{1+v^2}{4}\kappa^2\right]\delta'\left(p^2+\frac{1-v^2}{4}k^2+\kappa^2\right)e^{ikx}(dk)(dp)dv \qquad (3.38)$$

Comparing this with (3.26) and remembering that $D = \frac{3}{4}\Box^2 E$, we see that

$$D'(x) = -\frac{1}{4(2\pi)^7}\int_{-1}^{1}\iint\frac{1+v^2}{4}\,\delta'\left(p^2+\frac{1-v^2}{4}k^2+\kappa^2\right)e^{ikx}(dk)(dp)\,dv\,. \qquad (3.39)$$

Integrals (3.26) and (3.39) may be expanded by well-known methods. Each is equal to a logarthmically divergent multiple of the δ-function plus a finite part.

A final remark should be made concerning the sign of the induced energy density. The energy density in the meson field, i.e. the negative of the 4, 4 component of $\Theta_{M\mu\nu}$ in a locally Minkowskian coordinate system, should always be positive definite. Yet this is not, in general, true of the negative of the 4,4 component of the induced stress of (3.37). The induced energy density can have either sign. One is prompted to look for a resolution of this seeming paradox in the fourth order (and higher) calculations of the induced stress. As is well known, in the case of vacuum polarization in electrodynamics the fourth order induced charge has opposite sign to that of the second order, and so on, with signs alternating for each higher order. There is no compensation, since each infinity multiplies the preceding infinity, but the facts suggest that the notion of vacuum induced charge (and hence of vacuum induced stress) may become more intelligible on some non-perturbation basis; i.e. on a basis which embraces all approximation orders at once.

4. Gravitational Self-Energy of the Scalar Meson.

The gravitational self-energy of a scalar meson is readily calculated, to second order, from the interaction Hamiltonian density (2.17). One first observes that all first order processes are virtual. Even a transition in which a gravitational quantum annihilates with the creation of two other gravitational quanta, via the first order part of $\mathcal{H}_G^{SELF}[x,\sigma]$, can be a real transition only if all three quanta move in the same straight line. The density of final states for such processes is vanishingly small, and the processes are effectively virtual. The analytical expression of these statements is

$$\int_{-\infty}^{\infty} \left(\overset{1}{\mathcal{H}}_{INT}(x) + \overset{1}{\mathcal{H}}{}_G^{SELF}[x,\sigma]\right) d\omega = 0 \tag{4.1}$$

where $\overset{1}{\mathcal{H}}{}_G^{SELF}[x,\sigma]$ denotes the first order part of $\mathcal{H}_G^{SELF}[x,\sigma]$.

The first order interaction terms are transformed away by the canonical transformation

$$\left.\begin{aligned}
&\overline{\Psi}[\sigma] = e^{iS[\sigma]}\,\Psi[\sigma] \\
\text{where}\quad &S[\sigma] = \frac{1}{\hbar c}\int_{-\infty}^{\sigma} \left(\overset{1}{\mathcal{H}}_{INT}(x') + \overset{1}{\mathcal{H}}{}_G^{SELF}[x',\sigma]\right) d\omega' \\
&\quad\;\; = \frac{1}{2\hbar c}\int_{-\infty}^{\infty} \left(\overset{1}{\mathcal{H}}_{INT}(x') + \overset{1}{\mathcal{H}}{}_G^{SELF}[x',\sigma]\right) \varepsilon[\sigma,x']\, d\omega' \\
\text{satisfying}\quad &\hbar c\,\frac{\delta S[\sigma]}{\delta\sigma(x)} = \overset{1}{\mathcal{H}}_{INT}(x) + \overset{1}{\mathcal{H}}{}_G^{SELF}[x,\sigma]
\end{aligned}\right\} \tag{4.2}$$

[redacted]

The resulting second order interaction Hamiltonian density is given by

$$\overset{2}{\mathcal{H}}_{INT}[x,\sigma] = \overset{1}{\mathcal{H}}_{INT}(x) + \mathcal{H}_G^{SELF}[x,\sigma] + \overset{2}{\mathcal{H}}_{INT}[x,\sigma] + i\left[S[\sigma], \overset{1}{\mathcal{H}}_{INT}(x) + \overset{1}{\mathcal{H}}{}_G^{SELF}[x,\sigma]\right]$$

$$- \hbar c \frac{\delta S[\sigma]}{\delta\sigma(x)} - \frac{i\hbar c}{2}\left[S[\sigma], \frac{\delta S[\sigma]}{\delta\sigma(x)}\right]$$

$$= -\frac{i}{4\hbar c}\int\left[\overset{1}{\mathcal{H}}_{INT}(x) + \overset{1}{\mathcal{H}}{}_G^{SELF}[x,\sigma], \overset{1}{\mathcal{H}}_{INT}(x') + \overset{1}{\mathcal{H}}{}_G^{SELF}[x',\sigma']\right]\epsilon[\sigma,x']\,d\omega'$$

$$+ \overset{2}{\mathcal{H}}_{INT}[x,\sigma] + \overset{2}{\mathcal{H}}{}_G^{SELF}[x,\sigma]. \tag{4.3}$$

The only part of this interaction which is of interest here is that which is of the second order in the matter field variables. Denoting this by $\overset{2}{\mathcal{H}}_M[x,\sigma]$, we have

$$\overset{2}{\mathcal{H}}_M[x,\sigma] = -\frac{i}{16\hbar c}\int\Big(\tfrac{1}{2}[h_{\mu\nu}(x), h_{\sigma\tau}(x')]\{\overset{\circ}{\Theta}_{M\mu\nu}(x), \overset{\circ}{\Theta}_{M\sigma\tau}(x')\}$$

$$+ \tfrac{1}{2}[\overset{\circ}{\Theta}_{M\mu\nu}(x), \overset{\circ}{\Theta}_{M\sigma\tau}(x')]\{h_{\mu\nu}(x), h_{\sigma\tau}(x')\}\Big)\epsilon[\sigma,x']\,d\omega' \tag{4.4}$$

$$+ \overset{2}{\mathcal{H}}_{INT}[x,\sigma]$$

The one-meson part of this operator is the self-energy operator.

With the aid of the relations

$$\left.\begin{aligned}\langle\phi_{\mu\nu}\phi_{\sigma\tau}\rangle_0 &= \frac{\hbar\beta c}{2}(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\tau})D^{(1)}(0)\\ \langle\phi_{\mu\alpha}\phi_{\alpha\nu}\rangle_0 &= -2\langle\phi_{\mu\nu}\phi_{\alpha\alpha}\rangle_0 = 2\hbar\beta c\,\delta_{\mu\nu}D^{(1)}(0)\end{aligned}\right\} \tag{4.5}$$

the one-meson part of the operator $\overset{2}{\mathcal{H}}_{INT}[x,\sigma]$ is readily found to be given by

$$\overset{2}{\mathcal{H}}_{INT\;1,0}(x) = -\frac{\hbar\beta c}{4}D^{(1)}(0)\left(\phi_{,\mu}\phi_{,\mu} + 5\kappa^2\phi^2\right)_1(x), \tag{4.6}$$

in which the dependence on the surface normals disappears. The one-meson part of the integral in (4.4) has the form

$$\frac{\beta}{16}\int\Big[D(x-x')\Big(\{\overset{\circ}{\Theta}_{M\mu\nu}(x), \overset{\circ}{\Theta}_{M\mu\nu}(x')\}_1 - \tfrac{1}{2}\{\overset{\circ}{\Theta}_{M\mu\mu}(x), \overset{\circ}{\Theta}_{M\nu\nu}(x')\}_1\Big)$$

$$-i\,D^{(1)}(x-x')\Big([\overset{\circ}{\Theta}_{M\mu\nu}(x), \overset{\circ}{\Theta}_{M\mu\nu}(x')]_1 - \tfrac{1}{2}[\overset{\circ}{\Theta}_{M\mu\mu}(x), \overset{\circ}{\Theta}_{M\nu\nu}(x')]_1\Big)\Big]\epsilon[\sigma,x']\,d\omega'$$

With the use of equations (3.6), (3.7), (3.10) and (3.11), this expression can be reduced to

$$\frac{\hbar\beta c\kappa^2}{4}\iint\Big\{\bar{D}(x-x')\Big[\Delta^{(1)}_{,\alpha}(x-x')\big(\phi_{,\alpha}(x)\phi(x')\big)_1-\Delta^{(1)}_{,\alpha}(x-x')\big(\phi(x)\phi_{,\alpha}(x')\big)_1$$
$$+\Delta^{(1)}(x-x')\big(\phi_{,\alpha}(x)\phi_{,\alpha}(x')\big)_1+2\kappa^2\Delta^{(1)}(x-x')\big(\phi(x)\phi(x')\big)_1\Big]$$
$$+D^{(1)}(x-x')\Big[\bar{\Delta}_{,\alpha}(x-x')\big(\phi_{,\alpha}(x)\phi(x')\big)_1-\bar{\Delta}_{,\alpha}(x-x')\big(\phi(x)\phi_{,\alpha}(x')\big)_1$$
$$+\bar{\Delta}(x-x')\big(\phi_{,\alpha}(x)\phi_{,\alpha}(x')\big)_1+2\kappa^2\bar{\Delta}(x-x')\big(\phi(x)\phi(x')\big)_1\Big]\Big\}d\omega' \qquad (4.7)$$

which vanishes when $\kappa = 0$. In order to evaluate this integral we must consider the Fourier integral representations of the functions $\bar{D}\Delta^{(1)}+D^{(1)}\bar{\Delta}$ and $\bar{D}\Delta^{(1)}_{,\alpha}+D^{(1)}\bar{\Delta}_{,\alpha}$. For example, we have

$$\bar{D}(x)\Delta^{(1)}(x)+D^{(1)}(x)\bar{\Delta}(x)=\frac{1}{(2\pi)^7}\iint\Big[\frac{\delta(k'^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{k'^2+\kappa^2}\Big]e^{i(k+k')x}(dk)(dk')$$
$$=\frac{1}{(2\pi)^7}\iint\frac{\delta(k'^2+\kappa^2)-\delta(k^2)}{k^2-k'^2-\kappa^2}e^{i(k+k')x}(dk)(dk') \qquad (4.8)$$

If we introduce the integration variable $p = k + k'$ we notice that when (4.8) is substituted in (4.7) p is effectively restricted to $p^2+\kappa^2 = 0$ because of the wave equation satisfied by ϕ. p^2 may therefore be equivalently replaced by $-\kappa^2$. Denoting equivalence in this sense by the symbol $\doteq$, we may write

$$\bar{D}(x)\Delta^{(1)}(x)+D^{(1)}(x)\bar{\Delta}(x)\doteq\frac{1}{(2\pi)^7}\iint\frac{\delta(k^2-2pk)-\delta(k^2)}{2pk}e^{ipx}(dk)(dp)$$
$$=-\frac{1}{(2\pi)^7}\int_0^1\iint\delta'(k^2-2pku)e^{ipx}(dk)(dp)\,du$$
$$\doteq-\frac{1}{(2\pi)^7}\int_0^1\iint\delta'(q^2+\kappa^2u^2)e^{ipx}(dq)(dp)\,du$$
$$=\frac{1}{(2\pi)^2}\lim_{Q\to\infty}\log\frac{Q+\sqrt{Q^2+\kappa^2}}{\kappa}\,\delta(x), \qquad (4.9)$$

where $q = k - pu$. Similarly,

$$\bar{D}(x)\Delta^{(1)}_{,\alpha}(x)+D^{(1)}(x)\bar{\Delta}_{,\alpha}(x)\doteq\frac{1}{2(2\pi)^2}\Big(\lim_{Q\to\infty}\log\frac{Q+\sqrt{Q^2+\kappa^2}}{\kappa}+\frac{1}{2}\Big)\delta_{,\alpha}(x) \qquad (4.10)$$

and integral (4.7) reduces to

$$a(\phi_{,\mu}\phi_{,\mu}+\kappa^2\phi^2)_1(x)+\frac{\hbar\beta c\kappa^2}{64\pi^2}(\phi_{,\mu}\phi_{,\mu}-\kappa^2\phi^2)_1(x) \tag{4.11}$$

where $$a=\frac{3\hbar\beta c\kappa^2}{32\pi^2}\lim_{Q\to\infty}\log\frac{Q+\sqrt{Q^2+\kappa^2}}{\kappa} \tag{4.12}$$

The total self-energy operator now takes the form

$$\overset{2}{\mathcal{H}}_{M\,1,0}(x)=-\tfrac{1}{2}(A\phi_{,\mu}\phi_{,\mu}+B\kappa^2\phi^2)_1(x) \tag{4.13}$$

where

$$\left.\begin{aligned} A&=\frac{\hbar\beta c}{2}D^{(1)}(0)-2a-\frac{\hbar\beta c\kappa^2}{32\pi^2}\\ B&=\frac{5\hbar\beta c}{2}D^{(1)}(0)-2a+\frac{\hbar\beta c\kappa^2}{32\pi^2}\end{aligned}\right\} \tag{4.14}$$

This quadratically divergent self-energy operator can be transformed away, to the second order, by the canonical transformation

$$\overset{=}{\Psi}'[\sigma]=e^{iS[\sigma]}\overset{=}{\Psi}[\sigma] \quad \text{where} \quad S[\sigma]=\frac{1}{2\hbar c}\int\overset{2}{\mathcal{H}}_{M\,1,0}(x')\,\epsilon[\sigma,x']\,d\omega' \tag{4.15}$$

The transformed meson field variable is given, to second order, by

$$\begin{aligned}\phi'(x)&=\phi(x)-\frac{i}{2\hbar c}\int[\phi(x),\overset{2}{\mathcal{H}}_{M\,1,0}(x')]\,\epsilon[\sigma,x']\,d\omega'\\ &=\phi(x)+\frac{i}{4\hbar c}\int[\phi(x),A\phi_{,\mu}(x')\phi_{,\mu}(x')+B\kappa^2\phi(x')\phi(x')]\,\epsilon[\sigma,x']\,d\omega'\\ &=\phi(x)-\int[A\bar{\Delta}_{,\mu}(x-x')\phi_{,\mu}(x')-B\kappa^2\bar{\Delta}(x-x')\phi(x')]\,d\omega'\\ &=(1+A)\phi(x)+(B-A)\kappa^2\int\bar{\Delta}(x-x')\phi(x')\,d\omega'\end{aligned} \tag{4.16}$$

and satisfies the equation

$$\begin{aligned}\phi'_{,\mu\mu}(x)&=(1+2A-B)\kappa^2\phi(x)+(B-A)\kappa^4\int\bar{\Delta}(x-x')\phi(x')\,d\omega'\\ &=\kappa^2\phi'(x)+(A-B)\kappa^2\phi(x)\end{aligned} \tag{4.17}$$

Since, to zero order, $\phi(x)=\phi'(x)$, the following equation is exact to second order.

$$[\Box^2-(1+A-B)\kappa^2]\phi'(x)=0\,. \tag{4.18}$$

The self-energy operator thus produces a renormalization of the meson mass, the gravitational self-mass being given, to second order, by

$$\delta m = \tfrac{1}{2}(A-B)\,m\,. \tag{4.19}$$

It should now be noticed that a <u>different</u> result for the second order gravitational self-energy would have been obtained if we had chosen some tensor other than $h_{\mu\nu}$ as the fundamental gravitational field tensor. For example, we could equally well have chosen any tensor of the form

$$\psi_{\mu\nu} = h_{\mu\nu} + W h_{\mu\nu} h_{\alpha\alpha} + X h_{\mu\alpha} h_{\alpha\nu} + \delta_{\mu\nu}(Y h_{\alpha\alpha} h_{\beta\beta} + Z h_{\alpha\beta} h_{\alpha\beta}) + \cdots \tag{4.20}$$

which differs from $h_{\mu\nu}$ in the higher orders. This choice would not affect the first order coupling terms nor the properties of the free fields, but it would materially affect the second order coupling terms. The new second order term in the interaction Hamiltonian density would have the form

$$\overset{2}{\mathcal{H}}{}'_{INT}[x,\sigma] = \overset{2}{\mathcal{H}}_{INT}[x,\sigma] + \Delta\overset{2}{\mathcal{H}}_{INT}(x) \tag{4.21}$$

where $\overset{2}{\mathcal{H}}_{INT}[x,\sigma]$ is given by (2.19) but with $\psi_{\mu\nu}$ replacing $h_{\mu\nu}$, and where

$$\Delta\overset{2}{\mathcal{H}}_{INT} = \tfrac{1}{2}(W\psi_{\mu\nu}\psi_{\alpha\alpha} + X\psi_{\mu\alpha}\psi_{\alpha\nu} + Y\delta_{\mu\nu}\psi_{\alpha\alpha}\psi_{\beta\beta} + Z\delta_{\mu\nu}\psi_{\alpha\beta}\psi_{\alpha\beta})\,\mathring{\Theta}_{M\,\mu\nu} \tag{4.22}$$

The one-meson part of $\Delta\overset{2}{\mathcal{H}}_{INT}$ has the form

$$\Delta\overset{2}{\mathcal{H}}_{INT\ 1,0} = -\frac{\hbar\beta c}{2}(-W+2X-4Y+8Z)\,D^{(1)}(0)\,(\phi_{,\mu}\phi_{,\mu} + 2\kappa^2\phi^2)_1 \tag{4.23}$$

If, now, the coefficients W, X, Y, Z are chosen so as to satisfy $-W+2X-4Y+8Z = -2$, the one-meson part of $\overset{2}{\mathcal{H}}{}'_{INT}[x,\sigma]$ becomes

$$\overset{2}{\mathcal{H}}{}'_{INT\ 1,0}(x) = \frac{3\hbar\beta c}{4}\,D^{(1)}(0)\,(\phi_{,\mu}\phi_{,\mu} + \kappa^2\phi^2)_1(x) \tag{4.24}$$

and this operator would replace the operator of (4.6). The total self-energy operator would take the form (4.13), but with the coefficients A and B replaced by the coefficients

$$\left.\begin{aligned} A' &= -\frac{3\hbar\beta c}{2}D^{(1)}(0) - 2a - \frac{\hbar\beta c\kappa^2}{32\pi^2} \\ B' &= -\frac{3\hbar\beta c}{2}D^{(1)}(0) - 2a + \frac{\hbar\beta c\kappa^2}{32\pi^2} \end{aligned}\right\} \tag{4.25}$$

It is therefore evident that, by suitably choosing the fundamental tensor, the divergent part of the second order self-mass of the meson can be eliminated, leaving only a finite residue given by

$$\delta m = \tfrac{1}{2}(A' - B')\, m = -\frac{\hbar \beta c K^2}{32\pi^2}\, m = -\frac{G}{2\pi\hbar c}\, m^3 . \tag{4.26}$$

It may also be true that the divergent parts of higher order contributions to the self-mass can be eliminated in a similar fashion. One has a feeling, however, that any program which attempts to carry out such an elimination allows entirely too much flexibility to enter the theory and defeats the purpose of a perturbation calculation. On the other hand, the flexibility is actually in the theory to begin with. This may be made clear by an illustration.

Consider the simplest possible type of field, a massless scalar field, which has the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\psi_{,\mu}\psi_{,\mu} \tag{4.27}$$

Instead of regarding ψ as the fundamental field variable suppose we introduce a new variable ϕ related to ψ by

$$\phi = f^{-1}(\psi), \qquad \psi = f(\phi) \tag{4.28}$$

where f may be required to be an analytic function with f(0)=0, f'(0)=1. The Lagrangian function then takes the form

$$\mathcal{L} = -\tfrac{1}{2}[f'(\phi)]^2 \phi_{,\mu}\phi_{,\mu} \tag{4.29}$$

and, with respect to ϕ, the field equations are no longer linear. In fact, if one chooses the function f according to

$$f^{-1}(\psi) = \psi + \frac{\alpha}{2}\psi^2 \tag{4.30}$$

then the ϕ-field will satisfy a feedback principle based on the contracted stress tensor

$$\phi_{,\mu\mu} = -\alpha\,\Theta_{\mu\mu} = \alpha\,\psi_{,\mu}\psi_{,\mu} . \tag{4.31}$$

The ϕ-field experiences a self-interaction, and one may attempt to treat this self-interaction by perturbation methods. If a linear part is separated out of the Lagrangian (4.29) and a transition is made to the interaction representation, the interaction Hamiltonian density will be found to have the form

$$\mathcal{H}_{INT}[x,\sigma] = \frac{1}{2}\Big(n_\mu n_\nu\big[(f'(\phi))^{-2} + (f'(\phi))^2 - 2\big]\phi_{,\mu}\phi_{,\nu} + \big[(f'(\phi))^2 - 1\big]\phi_{,\mu}\phi_{,\mu}\Big)[x,\sigma] \qquad (4.32)$$

The coefficients in the brackets [ ] in this expression may be expanded in ascending powers of ϕ, and perturbation calculations based on powers of ϕ may be carried out. The self-energy operator, i.e. the one-particle part of (4.32), is immediately found to diverge to every order. This is not surprising, and we simply say to ourselves that we have chosen the wrong representation in which to carry out a perturbation calculation. A ϕ-quantum and a ψ-quantum are fundamentally different entities, and the ϕ-quantum is a false quantum. The correct representation is the linear ψ-representation (4.27).

In the case of the gravitational field, however, the non-linearity is essential, and there is no linear representation. With nothing to guide us, therefore, we are faced with unwanted flexibility and freedom in our choice of the fundamental tensor to be used in a perturbation theory. But we have just seen that this freedom may be partly curtailed by requiring that the second order meson self-mass be finite, and hence a program of divergence elimination may be just the thing we want. From the expansion (I, 5.7) it may be seen that if the contravariant metric tensor $g^{\mu\nu}$ had been chosen as the fundamental tensor instead of the covariant tensor $g_{\mu\nu}$, the second order self-mass would have taken on its unique finite value (4.26). Although this may be simply coincidence, it puts $g^{\mu\nu}$ forward as a candidate for "correct fundamental tensor." It would be of interest to see how $g^{\mu\nu}$ fares in a different setting, e.g. in the calculation of the gravitational self-mass of a Dirac particle.

5. The Electromagnetic Field.

The procedures outlined in the preceding sections may be applied almost without change to the problem of the electromagnetic field interacting with gravitation. The special points of interest in this problem have to do with the fact that the field equations are invariant under two different gauge groups. Electromagnetic gauge transformations will be referred to as E-gauge transformations in order to distinguish them from G-gauge transformations.

The series expansion of the total electromagnetic Lagrangian density has the form

$$\mathcal{L}_E = -\tfrac{1}{4} g^{\mu\sigma} g^{\nu\tau} F_{\mu\nu} F_{\sigma\tau} = \overset{0}{\mathcal{L}}_E + \overset{1}{\mathcal{L}}_E + \overset{2}{\mathcal{L}}_E + \cdots$$

where

$$\overset{0}{\mathcal{L}}_E = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{5.1}$$

$$\overset{1}{\mathcal{L}}_E = \tfrac{1}{2} h_{\mu\nu} \left(F_{\mu\alpha} F_{\nu\alpha} - \tfrac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} \right) = \tfrac{1}{2} h_{\mu\nu} \overset{0}{\Theta}_{E\,\mu\nu} \tag{5.2}$$

and

$$\overset{2}{\mathcal{L}}_E = -\tfrac{1}{4} \Big(h_{\mu\alpha} h_{\nu\beta} F_{\mu\nu} F_{\alpha\beta} - h_{\alpha\alpha} h_{\mu\nu} F_{\mu\beta} F_{\nu\beta} + 2 h_{\mu\alpha} h_{\alpha\nu} F_{\mu\beta} F_{\nu\beta} + \tfrac{1}{8} h_{\alpha\alpha} h_{\beta\beta} F_{\mu\nu} F_{\mu\nu} - \tfrac{1}{4} h_{\alpha\beta} h_{\alpha\beta} F_{\mu\nu} F_{\mu\nu} \Big) \tag{5.3}$$

As is well known, a Hamiltonian formulation of the free electromagnetic field equations cannot be carried out with the Lagrangian $\overset{0}{\mathcal{L}}_E$. It is conveniently replaced by

$$\hat{\mathcal{L}}_E = -\tfrac{1}{2} A_{\mu,\nu} A_{\mu,\nu} \tag{5.4}$$

together with the Lorentz supplementary condition. Accordingly, the Lagrangian function for the combined fields is taken in the form

$$\mathcal{L} = \hat{\mathcal{L}}_G + \hat{\mathcal{L}}_G^{SELF} + \hat{\mathcal{L}}_E + \overset{1}{\mathcal{L}}_E + \overset{2}{\mathcal{L}}_E + \cdots \tag{5.5}$$

together with the supplementary conditions

$$\left(\phi_{\mu\nu,\nu} + \cdots \right) \Phi = 0 \,, \tag{5.6}$$

and

$$A_{\mu,\mu} \Phi = 0 \,. \tag{5.7}$$

The interaction Hamiltonian density which is constructed from the Lagrangian (5.5) has the form (2.17) with

$$\overset{1}{\mathcal{H}}_{INT} = -\tfrac{1}{2} h_{\mu\nu} \overset{0}{\Theta}_{E\,\mu\nu} , \tag{5.8}$$

$$\overset{2}{\mathcal{H}}_{INT}[x,\sigma] = \Big[\tfrac{1}{2} n_\sigma n_\tau \big(h_{\mu\alpha} h_{\mu\beta} F_{\sigma\alpha} F_{\tau\beta} + h_{\sigma\alpha} h_{\tau\beta} F_{\alpha\mu} F_{\beta\mu} + \tfrac{1}{4} h_{\alpha\alpha} h_{\beta\beta} F_{\mu\sigma} F_{\mu\tau} - 2h_{\mu\alpha} h_{\tau\beta} F_{\alpha\sigma} F_{\beta\mu} - h_{\mu\alpha} h_{\beta\beta} F_{\alpha\sigma} F_{\mu\tau} + h_{\sigma\alpha} h_{\beta\beta} F_{\alpha\mu} F_{\mu\tau}\big) - \overset{2}{\mathcal{L}}_E\Big][x,\sigma], \quad (5.9)$$

and satisfies the integrability condition. It may also readily be shown to generate the correct electromagnetic field equations, to first order, in the Heisenberg representation.

In the interaction representation the supplementary condition (5.6) again takes the form (2.31), but with $\overset{\circ}{\Theta}_{M\mu\nu}$ replaced by $\overset{\circ}{\Theta}_{E\mu\nu}$. Because all the electromagnetic interaction terms $\overset{1}{\mathcal{H}}_{INT}(x)$, $\overset{2}{\mathcal{H}}_{INT}[x,\sigma]$,...etc. involve the electromagnetic potentials only through the E-gauge invariant field tensor $F_{\mu\nu}$, and because of the fact that $\int_\sigma [A_\mu(x'), F_{\nu\sigma}(x)]d\sigma'_\mu = -i\hbar c\int_\sigma [D_{,\nu}(x-x')d\sigma'_\sigma - D_{,\sigma}(x-x')d\sigma'_\nu]$ =0, it is found that $U[\sigma]A_{\mu,\mu}(x)U^*[\sigma] = A_{\mu,\mu}(x)$ and $U[\sigma]A_{\mu,\mu\nu}(x)U^*[\sigma] = A_{\mu,\mu\nu}(x)$, which means that supplementary condition (5.7) remains unchanged by the canonical transformation which leads to the interaction representation. That is,

$$A_{\mu,\mu}(x)\,\Psi[\sigma] = 0 \qquad (5.10)$$

whether x is contained in σ or not. Equation (5.10) is an exact equation, and expresses the fact that the gravitational field carries no charge.

G-gauge transformations require a special discussion in the present problem. The electromagnetic field equations in the Heisenberg representation have the following form to first order.

$$A_{\mu,\nu\nu} = -\big(h_{\mu\sigma} F_{\sigma\nu} - h_{\nu\sigma} F_{\sigma\mu} - \tfrac{1}{2} h_{\sigma\sigma} F_{\mu\nu}\big)_{,\nu} + \cdots \qquad (5.11)$$

Neither these equations nor the supplementary condition (5.7) remain invariant under the ordinary G-gauge transformation law

$$A'_\mu = A_\mu + A_{\mu,\sigma}\delta\Lambda_\sigma + A_\sigma \delta\Lambda_{\sigma,\mu} = A_\mu - F_{\mu\nu}\delta\Lambda_\nu + (A_\nu \delta\Lambda_\nu)_{,\mu} \qquad (5.12)$$

(cf. (I, 3.24)), and hence this law must be modified. The appropriate modification can be found by noting that equations (2.36) and (2.37) (with $\overset{\circ}{\Theta}_{M\mu\nu}$ replaced by $\overset{\circ}{\Theta}_{E\mu\nu}$) must continue to be valid in the present case, but that the first order solution (2,38) of equation (2.37) must be changed to the form

$$S[\sigma] = \frac{1}{\hbar c}\int_\sigma \overset{\circ}{\Theta}_{E\mu\nu}\delta\Lambda_\mu d\sigma_\nu + \frac{1}{2\hbar c}\int (F_{\mu\sigma} A_{\nu,\nu\sigma}\delta\Lambda_\mu)(x)\,\epsilon[\sigma,x]\,d\omega - \frac{1}{\hbar c}\int_{-\infty}^{\sigma} \delta \mathcal{H}_G^{SELF}[x,\sigma]\,d\omega + \cdots\cdots \qquad (5.13)$$

in which an extra term has been added in virtue of the fact that the divergence of $\overset{\circ}{\Theta}_{E\mu\nu}$ does not vanish but is equal to $-F_{\mu\sigma}A_{\nu,\nu\sigma}$. The canonical transformation defined by (5.13) induces a change in A_μ given, to first order, by

$$\delta A_\mu = -F_{\mu\nu}\delta\Lambda_\nu - \int\left[\bar{D}_{,\mu\nu}(x-x')F_{\nu\sigma}(x') + \bar{D}_{,\tau}(x-x')\left(\delta_{\tau\sigma}A_{\nu,\nu\mu} - \delta_{\mu\sigma}A_{\nu,\nu\tau}\right)(x')\right]\delta\Lambda_\sigma(x')\,d\omega' + \cdots \tag{5.14}$$

Condition (5.7) is seen to be unaffected by this change, i.e. $\delta A_{\mu,\mu} \equiv 0$. Moreover, using the fact that $\Box^2 A_\mu$ vanishes to zero order, we may write the change induced in the field equation (5.11) by a G-gauge transformation, to first order, in the form

$$\begin{aligned}
\delta A_{\mu,\nu\nu} &= -\left(F_{\sigma\nu}\delta\Lambda_{\mu,\sigma} + F_{\sigma\nu}\delta\Lambda_{\sigma,\mu} - F_{\sigma\mu}\delta\Lambda_{\nu,\sigma} - F_{\sigma\mu}\delta\Lambda_{\sigma,\nu} - F_{\mu\nu}\delta\Lambda_{\sigma,\sigma}\right)_{,\nu} + \cdots \\
&= -\left(-F_{\sigma\nu,\sigma}\delta\Lambda_\mu - F_{\sigma\nu,\mu}\delta\Lambda_\sigma + F_{\sigma\mu,\sigma}\delta\Lambda_\nu + F_{\sigma\mu,\nu}\delta\Lambda_\sigma + F_{\mu\nu,\sigma}\delta\Lambda_\sigma\right)_{,\nu} \\
&\quad - (F_{\sigma\nu}\delta\Lambda_\mu)_{,\sigma\nu} - (F_{\sigma\nu}\delta\Lambda_\sigma)_{,\mu\nu} + (F_{\sigma\mu}\delta\Lambda_\nu)_{,\sigma\nu} + (F_{\sigma\mu}\delta\Lambda_\sigma)_{,\nu\nu} + (F_{\mu\nu}\delta\Lambda_\sigma)_{,\sigma\nu} + \cdots \\
&= -(F_{\mu\nu}\delta\Lambda_\nu)_{,\alpha\alpha} + (F_{\nu\sigma}\delta\Lambda_\sigma)_{,\mu\nu} + (A_{\sigma,\sigma\mu}\delta\Lambda_\nu - A_{\sigma,\sigma\nu}\delta\Lambda_\mu)_{,\nu} + \cdots
\end{aligned} \tag{5.15}$$

which is clearly satisfied by (5.14). (5.14) therefore expresses the correct modified G-gauge transformation law for the electromagnetic potentials. It may be noted that in the classical form of the theory, in which the Lorentz condition is an identity rather than an eigenvalue equation, this modified law is obtained from the old law (5.12) simply by appending an <u>E-gauge transformation</u> generated by the scalar

$$\delta\Lambda(x) = -A_\nu(x)\delta\Lambda_\nu(x) - \int\bar{D}_{,\nu}(x-x')F_{\nu\sigma}(x')\delta\Lambda_\sigma(x')\,d\omega'. \tag{5.16}$$

That is, the Lorentz condition, which is invalidated by a G-gauge transformation, is restored by an appropriate E-gauge transformation. Equations (5.7) and (5.11) are, of course, invariant under any E-gauge transformation for which the gauge parameter satisfies the wave equation $\Box^2\Lambda = 0$, and we should also remember that the invariance of the gravitational field equations requires $\Box^2\delta\Lambda_\mu = 0$.

The procedure for eliminating the longitudinal components of the gravitational field in the electromagnetic case follows through exactly as in the meson case. The longitudinal electromagnetic field may, in virtue of (5.10), be eliminated at any stage in the game and the extra term dropped from (5.13). The Newtonian operator is given by (2.41), with $\overset{\circ}{\Theta}_{M\mu\nu}$ replaced by $\overset{\circ}{\Theta}_{E\mu\nu}$, and the static interaction energy is again expressed by (2.46). In this case, however, since the electromagnetic stress tensor is traceless, $\Theta_{ii} = U$, and (2.46) is modified to

$$-\frac{G}{c^4}\iint\left[\frac{7}{8}\frac{\{U',U\}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{c^2}\frac{\{\mathbf{S}_i',\mathbf{S}_i\}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{8}\frac{(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} : \{U',\boldsymbol{\Theta}\}\right] d\mathbf{r}\, d\mathbf{r}' \tag{5.17}$$

By evaluating the vacuum expectation value of the commutator[19] $[\mathring{\Theta}_{E\mu\nu}(x), \mathring{\Theta}_{E\sigma\tau}(x')]$

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[19] This expectation value is independent of whether the full electromagnetic potentials A_μ or the transverse potentials a_μ are used in the definition of $\mathring{\Theta}_{E\mu\nu}$.

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and carrying through a procedure identical with that outlined in section 3, one finds that the electromagnetic stress induced in the vacuum is given by

$$\langle \Theta_{E\mu\nu}[x,\sigma]\rangle = \frac{\hbar c}{2}\int G_{\mu\nu\sigma\tau}(x-x')\, h_{\sigma\tau}(x')\, d\omega' \tag{5.18}$$

with

$$\begin{aligned} G_{\mu\nu\sigma\tau} = {} & (\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\tau})\bar{D}_{,\alpha\beta}D^{(1)}_{,\alpha\beta} + 2\bar{D}_{,\mu\nu}D^{(1)}_{,\sigma\tau} + 2\bar{D}_{,\sigma\tau}D^{(1)}_{,\mu\nu} \\ & - \delta_{\mu\sigma}(\bar{D}_{,\alpha\nu}D^{(1)}_{,\alpha\tau} + \bar{D}_{,\alpha\tau}D^{(1)}_{,\alpha\nu}) - \delta_{\nu\tau}(\bar{D}_{,\alpha\mu}D^{(1)}_{,\alpha\sigma} + \bar{D}_{,\alpha\sigma}D^{(1)}_{,\alpha\mu}) \\ & - \delta_{\mu\tau}(\bar{D}_{,\alpha\nu}D^{(1)}_{,\alpha\sigma} + \bar{D}_{,\alpha\sigma}D^{(1)}_{,\alpha\nu}) - \delta_{\nu\sigma}(\bar{D}_{,\alpha\mu}D^{(1)}_{,\alpha\tau} + \bar{D}_{,\alpha\tau}D^{(1)}_{,\alpha\mu}) \\ & + \delta_{\mu\nu}(\bar{D}_{,\alpha\sigma}D^{(1)}_{,\alpha\tau} + \bar{D}_{,\alpha\tau}D^{(1)}_{,\alpha\sigma}) + \delta_{\sigma\tau}(\bar{D}_{,\alpha\mu}D^{(1)}_{,\alpha\nu} + \bar{D}_{,\alpha\nu}D^{(1)}_{,\alpha\mu}) \end{aligned} \tag{5.19}$$

G-gauge invariance and energy-momentum conservation now depend on the formal identification

$$D^{(1)}_{,\mu\nu\sigma}(0) = 0\,. \tag{5.20}$$

With the use of this identification and the relations $\Box^2\bar{D}(x) = -\delta(x)$, $\Box^2 D^{(1)} = 0$ and $D^{(1)}_{,\mu\nu}(0) = 0$, the divergence condition may be proved directly from (5.19).

$$G_{\mu\nu\sigma\tau,\tau} = 0\,. \tag{5.21}$$

Also,

$$G_{\mu\mu\sigma\tau} = 0\,. \tag{5.22}$$

Introducing the Fourier integral representation and making the change of variables (3.22), one finds that $G_{\mu\nu\sigma\tau}$ takes the form

$$\begin{aligned} G_{\mu\nu\sigma\tau} = {} & (\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma})G - \delta_{\mu\nu}\delta_{\sigma\tau}H - \delta_{\mu\sigma}J_{,\nu\tau} - \delta_{\nu\tau}J_{,\mu\sigma} - \delta_{\mu\tau}J_{,\nu\sigma} - \delta_{\nu\sigma}J_{,\mu\tau} \\ & + \delta_{\mu\nu}K_{,\sigma\tau} + \delta_{\sigma\tau}K_{,\mu\nu} + L_{,\mu\nu\sigma\tau} \end{aligned} \tag{5.23}$$

where

$$L(x) = -\frac{1}{4(2\pi)^7}\int_{-1}^{1}\iint\frac{(1-v^2)^2}{4}\,\delta'\!\left(p^2 + \frac{1-v^2}{4}k^2\right)e^{ikx}(dk)(dp)\,dv\,. \tag{5.24}$$

Reimposing relations (5.21) and (5.22) as conditions on the functions G, H, J, K, L, we have

$$0 = G_{\mu\nu\sigma\tau,\tau} = \delta_{\mu\sigma} G_{,\nu} + \delta_{\nu\sigma} G_{,\mu} - \delta_{\mu\nu} H_{,\sigma} - \delta_{\mu\sigma} J_{,\nu\tau\tau} - 2J_{,\mu\nu\sigma} - \delta_{\nu\sigma} J_{,\mu\tau\tau}$$
$$+ \delta_{\mu\nu} K_{,\sigma\tau\tau} + K_{,\mu\nu\sigma} + L_{,\mu\nu\sigma\tau\tau}$$

$$0 = G_{\mu\mu\sigma\tau} = 2\delta_{\sigma\tau} G - 4\delta_{\sigma\tau} H - 4J_{,\sigma\tau} + 4K_{,\sigma\tau} + \delta_{\sigma\tau} K_{,\mu\mu} + L_{,\mu\mu\sigma\tau}$$

leading to the equations

$$G - \Box^2 J = 0, \quad -H + \Box^2 K = 0, \quad -2J + K + \Box^2 L = 0, \quad 2G - 4H + \Box^2 K = 0, \quad -4J + 4K + \Box^2 L = 0,$$

of which the solutions are

$$\frac{4}{3} G = 2H = \frac{4}{3}\Box^2 J = 2\Box^2 K = \Box^4 L,$$

which enable us to write

$$G_{\mu\nu\sigma\tau} = \frac{3}{4}\Big[(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma}) L_{,\alpha\alpha\beta\beta} - \delta_{\mu\sigma} L_{,\alpha\alpha\nu\tau} - \delta_{\nu\tau} L_{,\alpha\alpha\mu\sigma} - \delta_{\mu\tau} L_{,\alpha\alpha\nu\sigma} - \delta_{\nu\sigma} L_{,\alpha\alpha\mu\tau}$$
$$+ 2\delta_{\sigma\tau} L_{,\alpha\alpha\mu\nu} + 2\delta_{\mu\nu} L_{,\alpha\alpha\sigma\tau} - 2\delta_{\mu\nu}\delta_{\sigma\tau} L_{,\alpha\alpha\beta\beta}\Big] + \Big[L_{,\mu\nu\sigma\tau} - \delta_{\sigma\tau} L_{,\alpha\alpha\mu\nu}\Big]$$
$$- \delta_{\mu\nu}\Big[L_{,\alpha\alpha\sigma\tau} - \delta_{\sigma\tau} L_{,\alpha\alpha\beta\beta}\Big] \tag{5.25}$$

and $$\langle \Theta_{E\mu\nu}[x,\sigma] \rangle = \frac{3\hbar c}{2} \int L(x-x') \Big(\overset{1}{\dot{G}}_{\mu\nu,\alpha\alpha} + \frac{1}{3} \overset{1}{\dot{G}}_{\alpha\alpha,\mu\nu} - \frac{1}{3}\delta_{\mu\nu} \overset{1}{\dot{G}}_{\alpha\alpha,\beta\beta}\Big)(x')\, d\omega'$$
$$= -\frac{3\hbar\beta c}{4} \int L(x-x') \Big(\overset{EXT}{\Theta}_{\mu\nu,\alpha\alpha} + \frac{1}{3} \overset{EXT}{\Theta}_{\alpha\alpha,\mu\nu} - \frac{1}{3}\delta_{\mu\nu} \overset{EXT}{\Theta}_{\alpha\alpha,\beta\beta}\Big)(x')\, d\omega'. \tag{5.26}$$

The kernel L(x) is identical with E(x) of (3.26) for the case $\kappa = 0$. It cannot be expanded by the usual methods, however, precisely because $\kappa = 0$. It can nevertheless be obtained in an explicit form. Carrying out a partial integration on (5.24), one finds

$$L(x) = \frac{1}{15(2\pi)^6} \int \lim_{\substack{P\to\infty \\ V\to 1}} \Big(\log \frac{2P}{\sqrt{|k^2|}} - \log \frac{\sqrt{|1-V^2|}}{2} - 1 \Big) e^{ikx} (dk)$$
$$- \frac{1}{16(2\pi)^6} \int_{-1}^{1} \Big(v - \frac{2v^3}{3} + \frac{v^5}{5} \Big) \frac{v}{1-v^2}\, dv \int e^{ikx} (dk)$$
$$= \frac{1}{15(2\pi)^6} \int \Big(\lim_{P\to\infty} \log 2P - \log\sqrt{|k^2|} - \frac{13}{60} \Big) e^{ikx} (dk) \tag{5.27}$$

Expression (5.27) is as far as one can go in the reduction of L(x) without specializing to particular cases. Using the equation

$$-\frac{1}{(2\pi)^3} \int \log |\mathbf{k}|\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} = \frac{1}{4\pi r^3} + (\gamma + 1)\,\delta(\mathbf{r}) + \log r\, \delta(\mathbf{r}) \tag{5.28}$$

(where $\gamma = .5772\ldots$) which may be verified by taking its Fourier transform, one may obtain the kernel which is useful in the case of time independent $\overset{EXT}{\Theta}_{\mu\nu}$, namely

$$L(\mathbf{r}) = \int_{-\infty}^{\infty} L(x)\, dx_0 = \frac{1}{60\pi^2} \Big(\lim_{P\to\infty} \log 2P + \log r + \gamma + 1 - \frac{13}{60} \Big) \delta(\mathbf{r}) + \frac{1}{240\pi^3 r^3}. \tag{5.29}$$

The gravitational self-energy of a photon may also be readily calculated. The history of this problem dates back to 1930, by which time the intensive program of investigating the divergence difficulties of quantum field theories had already begun. As one more proof of the untenability of existing theories, Rosenfeld[20] made a calculation which showed that the self-

[20] Zs. f. Phys. 65, 589 (1930).

energy of a photon due to its gravitational field diverges quadratically. Unfortunately, Rosenfeld's calculation, involving a problem in general relativity, was not even Lorentz invariant, let alone covariant under general coordinate transformations (i.e. G-gauge invariant). His calculation, moreover, made no use of E-gauge invariance, and, finally, he did not use the correct interaction Hamiltonian, choosing instead simply the negative of the 4,4 component of the interaction part of the canonical stress tensor in the Heisenberg representation and taking it over without change into the interaction representation, while at the same time ignoring second order terms. Our result will be considerably at variance with Rosenfeld's.

Following the procedure of section 4, we consider that part of the transformed interaction Hamiltonian density (4.3) which is of the second order in the electromagnetic field variables. Its one-photon part is given by

$$\overset{2}{\mathcal{H}}_{E\,1,0}(x) = \frac{\beta}{16}\int\Big[D(x-x')\{\mathring{\Theta}_{E\mu\nu}(x), \mathring{\Theta}_{E\mu\nu}(x')\}_1 - i D^{(1)}(x-x')[\mathring{\Theta}_{E\mu\nu}(x), \mathring{\Theta}_{E\mu\nu}(x')]_1\Big]\epsilon[\sigma,x']\,d\omega' + \overset{2}{\mathcal{H}}_{INT\,1,0}[x,\sigma] \tag{5.30}$$

We need here the readily verified relations

$$\{\mathring{\Theta}_{E\mu\nu}(x), \mathring{\Theta}_{E\mu\nu}(x')\}_1 = -4\hbar c\, D^{(1)}_{,\alpha\beta}(x-x')\big(F_{\alpha\gamma}(x)F_{\beta\gamma}(x')\big)_1 \tag{5.31}$$

$$[\mathring{\Theta}_{E\mu\nu}(x), \mathring{\Theta}_{E\mu\nu}(x')]_1 = -4i\hbar c\, D_{,\alpha\beta}(x-x')\big(F_{\alpha\gamma}(x)F_{\beta\gamma}(x')\big)_1 \tag{5.32}$$

$$\overset{2}{\mathcal{H}}_{INT\,1,0}[x,\sigma] = \tfrac{1}{2}\hbar\beta c\, D^{(1)}(0)\big(n_\mu n_\nu F_{\mu\alpha}F_{\nu\alpha} + \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}\big)_1[x,\sigma] \tag{5.33}$$

The term containing the surface normals in (5.33) is cancelled by a term in the integral which arises via relation (A.10) of the appendix. The remaining term in (5.33) may be absorbed into the integral by making use of the relation $\Box^2\bar{D}(x) = -\delta(x)$. We have

$$\overset{2}{\mathcal{H}}_{E\,1,0}(x) = \tfrac{1}{2}\hbar\beta c \int \left(\bar{D}\,\overset{(1)}{D}_{,\alpha\beta} - \tfrac{1}{4}\delta_{\alpha\beta}\bar{D}\,\overset{(1)}{D}_{,\mu\mu} + \overset{(1)}{D}\,\bar{D}_{,\alpha\beta} - \tfrac{1}{4}\delta_{\alpha\beta}\overset{(1)}{D}\,\bar{D}_{,\mu\mu}\right)(x-x')\left(F_{\alpha\gamma}(x)F_{\beta\gamma}(x')\right)_1 d\omega' \quad (5.34)$$

in which a vanishing term has been added for the sake of symmetry. Introducing the Fourier integral representation of the kernel in (5.34), changing the variables of integration to the variables p=k+k' and q=k-pu of section 4, and remembering that p is restricted to $p^2 = 0$ in virtue of the wave equation satisfied by $F_{\mu\nu}$, we find

$$\begin{aligned}
&\left(\bar{D}\,\overset{(1)}{D}_{,\alpha\beta} - \tfrac{1}{4}\delta_{\alpha\beta}\bar{D}\,\overset{(1)}{D}_{,\mu\mu} + \overset{(1)}{D}\,\bar{D}_{,\alpha\beta} - \tfrac{1}{4}\delta_{\alpha\beta}\overset{(1)}{D}\,\bar{D}_{,\mu\mu}\right)(x) \\
&= -\frac{1}{2(2\pi)^7}\iint\left(k'_\alpha k'_\beta - \tfrac{1}{4}\delta_{\alpha\beta}k'^2 + k_\alpha k_\beta - \tfrac{1}{4}\delta_{\alpha\beta}k^2\right)\frac{\delta(k'^2)-\delta(k^2)}{k^2-k'^2}\,e^{i(k+k')x}(dk)(dk') \\
&\doteq \frac{1}{2(2\pi)^7}\int_0^1\iint\left[p_\alpha p_\beta - p_\alpha k_\beta - p_\beta k_\alpha - \tfrac{1}{4}\delta_{\alpha\beta}(k^2-2pk) + 2k_\alpha k_\beta - \tfrac{1}{4}\delta_{\alpha\beta}k^2\right]\delta'(k^2-2pku)\,e^{ipx}(dk)(dp)\,du \\
&\doteq \frac{1}{2(2\pi)^7}\int_0^1\iint\left[p_\alpha p_\beta - p_\alpha(q_\beta+p_\beta u) - p_\beta(q_\alpha+p_\alpha u) - \tfrac{1}{4}\delta_{\alpha\beta}(q^2+2pqu-2pq)\right. \\
&\qquad\left. + 2(q_\alpha+p_\alpha u)(q_\beta+p_\beta u) - \tfrac{1}{4}\delta_{\alpha\beta}(q^2+2pqu)\right]\delta'(q^2)\,e^{ipx}(dq)(dp)\,du \\
&= \frac{1}{2(2\pi)^7}\int_0^1\iint(1-2u+2u^2)\,p_\alpha p_\beta\,\delta'(q^2)\,e^{ipx}(dq)(dp)\,du \\
&= \frac{1}{3(2\pi)^2}\lim_{Q\to\infty}\log 2Q\,\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\delta(x)
\end{aligned} \quad (5.35)$$

in which we have dropped the terms linear in q and have replaced $q_\alpha q_\beta$ by $\frac{1}{4}\delta_{\alpha\beta}q^2$ in passing to the next to the last expression. One immediately notices the remarkable fact that the terms in q^2 cancel each other, leaving only the terms in $p_\alpha p_\beta$. If one now substitutes (5.35) into (5.34), one finds

$$\overset{2}{\mathcal{H}}_{E\,1,0} = \frac{\hbar\beta c}{6(2\pi)^2}\lim_{Q\to\infty}\log 2Q\left(F_{\alpha\gamma}F_{\beta\gamma,\alpha\beta}\right)_1 \quad (5.36)$$

But $\quad F_{\alpha\gamma}F_{\beta\gamma,\alpha\beta} = \frac{1}{2}F_{\alpha\gamma}(F_{\beta\gamma,\alpha\beta} - F_{\beta\alpha,\gamma\beta}) = \frac{1}{2}F_{\alpha\gamma}F_{\alpha\gamma,\beta\beta} = 0.$

Therefore, the gravitational self-energy of a photon, to second order, is identically zero.

This completely unexpected result is entirely independent of what tensor we may choose as the fundamental tensor of the gravitational field. Since the trace of the electromagnetic stress tensor vanishes, a transformation of field variables of the form (4.19) will have no effect on the second order self-energy. The only other field whose stress tensor is traceless is the Dirac field for zero mass. It would be of interest, therefore, to carry out a self-energy calculation for this field also.

In conclusion, the author wishes to express his appreciation to Professor Schwinger for encouragement in carrying out this work, and to Professor Oppenheimer, Dr. A. Pais and Dr. M. Neuman for helpful discussions.

Appendix.

I. Theorems involving space-like surfaces and the invariant δ-functions.

A family of space-like surfaces may be defined by an equation of the form

$$\phi = \text{constant}, \tag{A.1}$$

the unit normal vectors being then given by

$$n_\mu = \phi_{,\mu}(-\phi_{,\nu}\phi_{,\nu})^{-\frac{1}{2}}. \tag{A.2}$$

The tangential derivative of the unit normal vector at any point takes the form

$$\begin{aligned}\partial_\nu n_\mu &= n_{\mu,\nu} + n_\nu n_\sigma n_{\mu,\sigma} \\ &= \phi_{,\mu\nu}(-\phi_{,\sigma}\phi_{,\sigma})^{-\frac{1}{2}} + (\phi_{,\mu}\phi_{,\sigma}\phi_{,\sigma\nu} + \phi_{,\nu}\phi_{,\sigma}\phi_{,\sigma\mu})(-\phi_{,\tau}\phi_{,\tau})^{-\frac{3}{2}} \\ &\quad + \phi_{,\mu}\phi_{,\nu}\phi_{,\sigma}\phi_{,\tau}\phi_{,\sigma\tau}(-\phi_{,\alpha}\phi_{,\alpha})^{-\frac{5}{2}}.\end{aligned}$$

This last expression, being symmetric in μ and ν, implies

$$\partial_\nu n_\mu = \partial_\mu n_\nu \tag{A.3}$$

Equation (A.3), involving, as it does, only tangential derivatives, does not really depend on the definition of n_μ in terms of a unit vector field as in (A.2). Its validity depends on the properties of none of the surfaces except the one which happens to pass through the point in question, and it merely expresses the fact that this surface is smooth (i.e. differentiable).

For functions f which vanish sufficiently rapidly in space-like directions, the theorem $\int_\sigma (f_{,\nu}d\sigma_\mu - f_{,\mu}d\sigma_\nu) = 0$, which Schwinger has proved (Phys. Rev. 74, 1439 (1948), eq. 1.58), really involves only tangential derivatives and may equally well be written in the form

$$\int_\sigma (n_\mu\partial_\nu - n_\nu\partial_\mu) f\, d\sigma = 0. \tag{A.4}$$

(A.4), together with the identity $n_\nu \partial_\mu n_\nu \equiv 0$, may be used to prove the following useful theorem.

$$\int_\sigma \partial_\mu f \, d\sigma = -\int_\sigma n_\nu n_\nu \partial_\mu f \, d\sigma = -\int_\sigma n_\nu \partial_\mu (n_\nu f) \, d\sigma = -\int_\sigma n_\mu \partial_\nu (n_\nu f) d\sigma$$
$$= -\int_\sigma n_\mu (\partial_\nu n_\nu) f \, d\sigma = -\int_\sigma k f \, d\sigma_\mu \tag{A.5}$$

where $k \equiv \partial_\mu n_\mu$. k is sometimes called the mean curvature of the surface at the point in question. For a flat surface the integral (A.5) vanishes.

The invariant delta function $\Delta(x)$ is defined by the equations

$$(\Box^2 - \kappa^2)\Delta = 0, \qquad \int_\sigma f \Delta \, d\sigma = 0, \qquad \int_\sigma f \frac{\partial \Delta}{\partial n} d\sigma = -f(0) \tag{A.6}$$

where f(x) is an arbitrary function and σ is any space-like surface passing through the origin. Denoting the functional dependence of the normal n_μ on the surface σ at the point x explicitly by $n_\mu[x,\sigma]$, we may write the following equations

$$\int_\sigma f \Delta_{,\mu} d\sigma = \int_\sigma f\left(\partial_\mu \Delta + n_\mu \frac{\partial \Delta}{\partial n}\right) d\sigma = \int_\sigma \left[\partial_\mu (f\Delta) - (\partial_\mu f)\Delta + f n_\mu \frac{\partial \Delta}{\partial n}\right] d\sigma$$
$$= \int_\sigma \left[(-k f n_\mu - \partial_\mu f)\Delta + f n_\mu \frac{\partial \Delta}{\partial n}\right] d\sigma = -n_\mu[0,\sigma] f(0) \tag{A.7}$$

$$\int_\sigma f \Delta_{,\mu\nu} d\sigma = \int_\sigma [(f\Delta_{,\mu})_{,\nu} - f_{,\nu}\Delta_{,\mu}] d\sigma$$
$$= n_\mu[0,\sigma] f_{,\nu}(0) + \int_\sigma [\partial_\nu (f\Delta_{,\mu}) - n_\nu n_\sigma (f\Delta_{,\mu})_{,\sigma}] d\sigma$$
$$= n_\mu[0,\sigma] f_{,\nu}(0) + \int_\sigma [-k n_\nu f \Delta_{,\mu} - n_\nu n_\sigma f_{,\sigma} \Delta_{,\mu} - n_\nu n_\sigma f \Delta_{,\mu\sigma}] d\sigma$$
$$= (k n_\mu n_\nu f + n_\mu \partial_\nu f)[0,\sigma] - \int_\sigma [n_\sigma (n_\nu f \Delta_{,\sigma})_{,\mu} - n_\sigma (n_\nu f)_{,\mu} \Delta_{,\sigma}] d\sigma$$
$$= [k n_\mu n_\nu f + n_\mu \partial_\nu f + (n_\nu f)_{,\mu}][0,\sigma] - \int_\sigma n_\mu (n_\nu f \Delta_{,\sigma})_{,\sigma} d\sigma$$
$$= [k n_\mu n_\nu f + n_\mu \partial_\nu f + (n_\nu f)_{,\mu}][0,\sigma] - \int_\sigma [n_\mu (n_\nu f)_{,\sigma} \Delta_{,\sigma} + \kappa^2 n_\mu n_\nu f \Delta] d\sigma$$
$$= [(k n_\mu n_\nu + \partial_\mu n_\nu) f + (n_\mu \partial_\nu + n_\nu \partial_\mu) f][0,\sigma] \tag{A.8}$$

In the derivation of (A.8) the normal n_μ is assumed to be defined, as in (A.2), in terms of a unit vector field. The final expression, however, contains only tangential derivatives and therefore depends only on the properties of the normals n_μ and the function f on the surface σ itself.

The function $\bar{\Delta}(x)$ is defined in terms of the functional $\varepsilon[\sigma, x]$, which is equal to +1 or -1 according as the point x is to the past or to the future of a space-like surface σ passing through the origin, by the equation

$$\bar{\Delta}(x) = \tfrac{1}{2}\,\varepsilon[\sigma, x]\,\Delta(x) \tag{A.9}$$

$\bar{\Delta}(x)$ satisfies the equation $\bar{\Delta}_{,\mu}(x) = \frac{1}{2}\varepsilon[\sigma,x]\Delta_{,\mu}(x)$. Also, for any function f(x),

$$\int_\infty \bar{\Delta}_{,\mu\nu}(x) f(x)\, d\omega = -\int_\infty \bar{\Delta}_{,\mu}(x) f_{,\nu}(x)\, d\omega = -\tfrac{1}{2}\int_\infty \varepsilon[\sigma,x]\,\Delta_{,\mu}(x) f_{,\nu}(x)\, d\omega$$

$$= -\int_\sigma \Delta_{,\mu}(x) f(x)\, d\sigma_\nu + \tfrac{1}{2}\int_{-\infty}^{\sigma} \Delta_{,\mu\nu}(x) f(x)\, d\omega - \tfrac{1}{2}\int_\sigma^{\infty} \Delta_{,\mu\nu}(x) f(x)\, d\omega$$

$$= n_\mu[0,\sigma]\, n_\nu[0,\sigma]\, f(0) + \tfrac{1}{2}\int_\infty \varepsilon[\sigma,x]\,\Delta_{,\mu\nu}(x) f(x)\, d\omega$$

which implies

$$\varepsilon[\sigma,x]\,\Delta_{,\mu\nu}(x) = 2\bar{\Delta}_{,\mu\nu}(x) - 2n_\mu[0,\sigma]\, n_\nu[0,\sigma]\,\delta(x) \tag{A.10}$$

The equation $(\Box^2 - \kappa^2)\bar{\Delta}(x) = -\delta(x)$ is a corollary of (A.10).

Although one may say $\Delta(0) = 0$, and also $\Delta(x) = 0$ and $\Delta_{,\mu}(x) = 0$ for $x_\mu x_\mu > 0$, one is not permitted to say $\Delta_{,\mu\nu}(x) = 0$ at points in the immediate space-like vicinity of the origin. Expressions involving $\Delta_{,\mu\nu}(x)$ with $x_\mu x_\mu > 0$ must be left standing as they are. Variational derivatives of products of surface normals involve such expressions. One observes that

$$n_\mu[x';\sigma]\, n_\nu[x';\sigma]\, f(x') = -\int_\sigma f(x)\,\Delta_{,\nu}(x-x')\, d\sigma_\mu = -f(x')\int_\sigma \Delta_{,\nu}(x-x')\, d\sigma_\mu$$

For $x \neq x'$ [redacted] (and hence $(x_\mu - x'_\mu)^2 > 0$), one may evidently write

$$\frac{\delta}{\delta\sigma(x)}\left(n_\mu[x';\sigma]\, n_\nu[x';\sigma]\, f(x')\right) = -f(x')\,\Delta_{,\mu\nu}(x-x') = -f(x)\,\Delta_{,\mu\nu}(x-x'). \tag{A.11}$$

II. Theorems involving the interaction Hamiltonian density.

Schwinger has shown (Phys. Rev. 74, 1439 (1948), eq. 2.9) how to express the first derivatives of the interaction field variables in terms of those of the Heisenberg field variables.

$$Q_{i,\mu}(x) = \frac{\delta}{\delta\sigma(x)}\int_\sigma Q_i(x')\, d\sigma'_\mu = \frac{\delta}{\delta\sigma(x)}\int_\sigma U[\sigma]\, Q_i(x')\, U^*[\sigma]\, d\sigma'_\mu$$

$$= U[\sigma]\, Q_{i,\mu}(x)\, U^*[\sigma] - \frac{1}{i\hbar c}\int_\sigma [Q_i(x'), \mathcal{H}_{INT}[x,\sigma]]\, d\sigma'_\mu \tag{A.12}$$

A corresponding expression for the second derivatives is obtained in a similar manner

$$Q_{i,\mu\nu}(x) = \frac{\delta}{\delta\sigma(x)} \int_\sigma Q_{i,\mu}(x')\, d\sigma_\nu'$$

$$= \frac{\delta}{\delta\sigma(x)} \int_\sigma \Big(U[\sigma]\, Q_{i,\mu}(x')\, U^*[\sigma] - \frac{1}{i\hbar c} \int_\sigma [Q_i(x''), \mathcal{H}_{INT}[x';\sigma]]\, d\sigma_\mu'' \Big)\, d\sigma_\nu'$$

$$= U[\sigma]\, Q_{i,\mu\nu}(x)\, U^*[\sigma] - \frac{1}{i\hbar c} \int_\sigma \Big([Q_{i,\mu}(x'), \mathcal{H}_{INT}[x,\sigma]] + [Q_{i,\mu}(x), \mathcal{H}_{INT}[x';\sigma]] \Big)\, d\sigma_\nu'$$

$$- \frac{1}{i\hbar c} \int_\sigma [Q_i(x'), \mathcal{H}_{INT,\nu}[x,\sigma]]\, d\sigma_\mu' - \frac{1}{i\hbar c} \int_\sigma \int_\sigma [Q_i(x''), \frac{\delta}{\delta\sigma(x)} \mathcal{H}_{INT}[x';\sigma]]\, d\sigma_\mu''\, d\sigma_\nu'$$

$$+ \frac{1}{\hbar^2 c^2} \int_\sigma \int_\sigma \Big[[Q_i(x''), \mathcal{H}_{INT}[x';\sigma]], \mathcal{H}_{INT}[x,\sigma] \Big]\, d\sigma_\mu''\, d\sigma_\nu' \qquad \text{(A.13)}$$

III. Construction of the functions $\overset{1}{\mathcal{D}}$ and $\overset{2}{\mathcal{D}}$.

In a coordinate system in which n_μ takes the form (0,0,0,-i), equations (1.20) take the form

$$D = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \overset{1}{\mathcal{D}} = \nabla^2 \overset{1}{\mathcal{D}}, \qquad \overset{1}{\mathcal{D}} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \overset{2}{\mathcal{D}} = \nabla^2 \overset{2}{\mathcal{D}} \; . \qquad \text{(A.14)}$$

According to Schwinger (Phys, Rev. 75, 651 (1949), eq. A.23), the function D(x) is given by

$$D(x) = \frac{1}{2\pi} \varepsilon[\sigma, x]\, \delta(x_\mu x_\mu) = -\frac{1}{2\pi} \frac{t}{|t|} \delta(r^2 - c^2 t^2) \qquad \text{(A.15)}$$

Hence
$$\overset{1}{\mathcal{D}}(\mathbf{r}, t) = -\frac{1}{4\pi} \int \frac{D(\mathbf{r}', t)}{|\mathbf{r}' - \mathbf{r}|}\, d\mathbf{r}' = \frac{1}{8\pi^2} \frac{t}{|t|} \int \frac{\delta(r'^2 - c^2t^2)}{|\mathbf{r}' - \mathbf{r}|}\, d\mathbf{r}'$$

$$= \begin{cases} \frac{1}{4\pi} \frac{t}{|t|} & r < c|t| \\ \frac{c}{4\pi} \frac{t}{r} & r > c|t| \; . \end{cases} \qquad \text{(A.16)}$$

By direct differentiation, one may next verify that

$$\overset{2}{\mathcal{D}}(\mathbf{r}, t) = \begin{cases} \frac{1}{24\pi} (r^2 + 3c^2t^2) \frac{t}{|t|} & r < c|t| \\ \frac{c}{24\pi} (3r^2 + c^2t^2) \frac{t}{r} & r > c|t| \end{cases} \qquad \text{(A.17)}$$

The time derivatives of these functions at t = 0 are of some importance.

$$\dot{\overset{1}{\mathcal{D}}}(\mathbf{r}, 0) = \frac{c}{4\pi} \frac{1}{r}, \qquad \dot{\overset{2}{\mathcal{D}}}(\mathbf{r}, 0) = \frac{c}{8\pi} r \; . \qquad \text{(A.18)}$$